Existence and Nonexistence Theorems for Solutions to the Einstein-Dirac-Maxwell Equations

Jeffrey Morton

Department of Mathematics and Statistics

McGill University, Montreal

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Dedication

"Just because they aren't real doesn't mean they don't exist."

This Work is dedicated to Them.

Thanks go to my supervisor, Niky Kamran; to my parents; to my friends who have supported me through the writing of this thesis - Jason Parent, Freida Abtan, Denise Robinson, Olivia Li, Andrew Archibald, Floyd Gecko, Lisa Toth, and many others.

The soundtrack to the writing of this thesis:

Going Well Spacetime Continuum, Pole, :zoviet*france:, Main, Coil, Gas, Mark Spybey, James Plotkin, Phil Western, Download, Autechre, Oval, Mick Harris, MBM, Plateau, Crawl Unit, Plastikman, David Kristian, Not Breathing, Legendary Pink Dots

Not So Well FLA, 242, Skinny Puppy, Throbbing Gristle, Merzbow, Phÿcus, Young Gods, C-Tec

Abstract

This thesis deals with representative examples from a recent body of work dealing with coupling of Einsteinian gravity to quantum-mechanical matter fields, and in particular the Dirac field and the electromagnetic field. The first part of the thesis develops the proof of the existence of particle-like (soliton) solutions to the fully coupled Einstein-Dirac equation, from the derivation of the form of the equations and their numerical solution to a numerical and topological analysis of the stability of the solutions found. In the second half of the thesis, two nonexistence theorems are developed for black-hole solutions for the Einstein-Dirac-Maxwell system with various coupling-strengths and symmetry conditions. These nonexistence theorems show the impossibility of stable, nontrivial, Dirac fields in the presence of black holes in the cases investigated.

Cette thèse traite des exemples représentatifs d'oeuvres récents concernant la conjoncture de la gravitation einsteinienne avec les champs de matière quantizées, notamment le champs de Dirac et de l'électromagnétisme. Sa première partie développe la preuve de l'existence des solutions reliées aux particules (soliton) à l'équation Einstein-Dirac, à partir de la dérivation de la forme des équations et leur solution numérique jusqu'à l'analyse numérique et topologique de la stabilité des solutions trouvées. Dans la deuxième partie de la thèse, on développe deux théorèmes de nonexistence aux solutions de "trous noirs" pour le système des équations Einstein-Dirac-Maxwell avec divers forces d'accouplement et conditions de symétrie. Ces théorèmes de nonexistence démontrent l'impossibilité d'un champ de Dirac stable et nonnul dans la présence de trous noirs dans les situations examinées.

Contents

Chapter 1

Introduction

1.1 Preamble

The purpose of this thesis is to examine and collect some recent research results concerning solutions to the Einstein-Dirac-Maxwell (EDM) system of equations. The results we examine are due to Finster, Kamran, Smoller and Yau, and appear in [FSY1], [FSY2] and [FKSY1]. These equations govern the behaviour of Dirac particles (such as electrons or neutrinos) coupled to gravity and electromagnetism. The results we shall be concerned with fall into two classes: there are results showing the existence of particle-like solutions to these equations; and there are results showing nonexistence of certain classes of singular solutions - in particular, black-hole solutions with various types of symmetry. It should be noted that not all of these results apply to the fully coupled EDM system.

This work is divided into two main sections. In the first section, we examine the results of [FSY1] regarding the existence of soliton (particle-like) solutions for the Einstein-Dirac equation in the static, spherically symmetric case. These results confirm that, for minimal coupling to electromagnetism, the equations governing the Dirac field do indeed predict the existence of particle-like solutions. In the second section, we examine two main results. The first is from [FSY2], dealing with the nonexistence of time-periodic solutions of the Dirac equation in the fairly general case of an axisymmetric black-hole background; that is, the (somewhat surprising) nonexistence of solutions in which a Dirac particle "orbits" such a black hole. The second result of the third section demonstrates the nonexistence of a class of static, spherically symmetric solutions to the full EDM system. There are two appendices giving additional background necessary to the main matter but which is not the principal subject of this work. In the first appendix we briefly give some necessary background material to define the Einstein, Dirac, and Maxwell equations and their setting. In the second we give some exposition regarding the use of topological methods, and in particular the Conley index theory, in qualitative analysis of PDEs, since this method is used in examining the stability of the particle-like solutions found in the first part.

1.2 Background of the Work

The work examined here is a recent selection of work in the subject of the coupling of gravity to various other fields. The initial work most closely related to the results we shall be dealing with was the work of Bartnik and McKinnon (1988), who studied the interaction of Einsteinian gravity with a non-Abelian Yang-Mills field. Their discovery of nontrivial particle-like solutions to the EYM system was somewhat remarkable, since these solutions were everywhere regular and static. Neither the Einstein vacuum equations nor the Yang-Mills equations uncoupled to gravity admit static, regular solutions. This fact is due to the presence of two forces, the repulsive Yang-Mills force and the attractive gravitational force, in the system which balance each other for the solutions found. Prior to this result, it had been conjectured that no such solutions could exist - their discovery was the first in a substantial body of recent work. The Bartnik-McKinnon solutions were shown by Straumann and Zhou (1990) to be unstable with respect to small perturbations - which introduces the theme of stability analysis for particle-like solutions, which shall be relevant to the current results. Substantial research on the Einstein-Yang-Mills equations has been done by Künzle and others (especially Darian and Masood-ul-Alam), particularly in finding cosmological solutions, and solutions with spherical symmetry. Further work with the EYM equations was undertaken by McLeod, Smoller, Wasserman, and Yau (1991) and by Smoller, Wasserman and Yau (1993), introducing in this original context the study of black hole solutions. They demonstrated the existence of such solutions, establishing a nontrivial class of black holes with Yang-Mills field. Smoller and Wasserman (1993) established the existence of infinitely many smooth solutions of the EYM equations.

These results with the non-Abelian Yang-Mills gauge fields set the stage for later work which examined similar questions about the Dirac field. Since the Dirac field represents fermions, which constitute normal matter, this is physically significant, but is greatly different in character from the Yang-Mills field. Substantial work with forms of the Einstein-Dirac and Einstein-Dirac-Maxwell equations has been done by Finster, Smoller, and Yau. The question of finding particle-like solutions for the Einstein-Dirac equation (studied in some depth in this thesis) was followed by a similar result for the full Einstein-Dirac-Maxwell system - in each case, soliton solutions were found, and shown in addition to be stable under small spherically symmetric perturbations. With these existence results, it is then natural to examine the case of black-hole solutions.

Unlike the Yang-Mills situation, the principal results for black-hole solutions are nonexistence theorems. In a series of papers from 1999, Finster, Smoller and Yau prove a number of related nonexistence theorems, relying on a few basic techniques well illustrated by the examples we study in detail in later parts of this work. The first, proving nonexistence of of static, spherically symmetric solutions to the fully coupled EDM system, relies on an analytic result, establishing bounds on the magnitude of the Dirac spinors which lead to contradiction for nonzero fields, meaning that only the Reissner-Nordström and Schwarzchild solutions are possible. The combination of spin and quantization changes the situation from the classical (nonquantum) picture. Turning to the minimally coupled situation, then, they looked at the Reissner-Nordström background and looked at the behaviour of a Dirac field uncoupled to gravity on this background. For this, because of the timelike nature of the singularity and the fact that the maximal analytic extension of the Reissner-Nordström background has infinitely many asymptotically flat regions connected through the black hole, it was necessary to develop matching conditions across the event horizon. This paper thus brought to the work a novel treatment of the Dirac equation in the distributional sense, seeking generalized solutions (which would be problematic in the fully coupled case since we assume regularity of the metric). The main result was to show that there are no time-periodic (and hence no static) solutions to the Dirac equation on this background.

This was generalized by Finster, Kamran, Smoller and Yau (in the last result studied in the present work) to the Kerr-Newman geometry (the most general Einstein-Maxwell black hole geometry), again using matching conditions and a distributional understanding of solutions of the Dirac equation. As in the former case, however, it was shown that there are no nontrivial solutions. This paper ([FKSY1]) also shows this result for more general geometries (a case not pursued in the present work) in which the Dirac equation is separable into radial and angular parts - namely geometries in which the Weyl conformal curvature tensor satisfies an algebraic condition making it "Type D" (a more general type of metric which includes, in addition to the KN geometry, others such as the Taub-NUT metric). This illustrates the application of the algebraic classification of the conformal curvature to show such general results. These nonexistence results for time-periodic solutions led to the investigation of the long-time dynamical behaviour of Dirac fields on these backgrounds given initial data. This too has been studied by Finster, Kamran, Smoller and Yau ([FKSY2]), and bounds have been found on the rate at which Dirac particles must escape to infinity or fall into the black hole.

Returning again to the Yang-Mills field whose coupling to gravitation began our discussion of the research in this area, an examination of the case of a Dirac particle coupled both to gravity and to the magnetic component of an $SU(2)$ Yang-Mills field - the Einstein-Dirac-Yang-Mills equation - has been done by Finster, Smoller and Yau. It was shown that the only solutions are the known black hole solutions with vanishing Dirac field. This makes use of a similar analytic approach to that seen in the spherically symmetric EDM case, by deriving bounds on the spinors at the horizon.

These recent developments in this area employ a wide range of techniques, most of which are exemplified by the particular cases studied in the current work.

1.3 Results to be Considered

We have now framed the problem to be considered: the interaction of three fields, namely the gravitational, electromagnetic and Dirac fields, and the solutions to the coupled systems of equations representing them. None of the solutions we will present are fully general, but each sheds some light on the more general question of classifying these solutions. In the next chapter, we begin with the positive result of the existence of particle-like solutions to the Einstein-Dirac equation. We suppress the electromagnetic field interactions (that is, we assume there is no electromagnetic field, so that we are dealing with a bare chargeless Dirac field coupled to gravity). This corresponds to the situation where gravitation is the dominant effect: it serves as a model problem for the more general, physically realistic case.

In the first section, we seek soliton solutions to the E-D equation. These are solutions which resemble particles in that they are locally concentrated, and spacetime is asymptotically flat: we seek these by use of a particular form for the field having this property. We assume such solutions to be spherically symmetric, and to follow a particular ansatz (this is admittedly not an entirely general approach, but since we are seeking an existence proof, the only work justification of the ansatz which is required is that is yield solutions to the equations). We then examine the stability of the solutions found. Establishing the existence and stability of particle-like solutions for this E-D system is rather difficult, and involves algebraic manipulation of spinorial and tensorial equations to obtain the form of the system to be studied, numerical computations to find solutions to the differential equations thus obtained, and topological analysis of these solutions to establish their stability.

In the second section, two nonexistence theorems are developed, generalizing somewhat the classification theorems of Carter, Israel and Robinson to include the possible presence of a nontrivial Dirac field. The first result addresses the spherically symmetric case only, with full coupling of the Dirac field to the metric. The second deals with the case of no coupling, describing a Dirac field on an axisymmetric background somewhat more general than the Kerr-Newman. It has been shown by Chandrasekhar that the Dirac equation is separable into ordinary differential equations in the Kerr-Newman background geometry, which makes possible the result of [FKSY1] (when suitably generalized). This can be regarded as an approximation of the weak-coupling limit for the full Einstein-Dirac-Maxwell equation.

Part I

Existence of Particle-Like Solutions of the Einstein-Dirac Equation

Chapter 2

Dirac Equation on a Static, Spherically Symmetric Background

2.1 Form of the Equations

2.1.1 Form of the Operator

We wish to find particle-like solutions of the Einstein-Dirac equation: we must thus find a metric on a manifold, and a corresponding Dirac field so that the stress-energy tensor associated with the field satisfies the Einstein field equations with the given metric and the field itself is a solution to the Dirac equation on that background. This requires the solution of a coupled set of equations: we must concretely find these equations and attempt to find solutions for them. This system is quite complicated, however, so to simplify the form as much as possible, we begin by assuming a highly symmetrical form to the spacetime. In particular, we shall assume a spherically symmetric spacetime. A common example of such a spacetime is the Schwarzchild solution, which in spherical coordinates (T, r, θ, ϕ), has metric form

$$\mathbf{ds}^2 = -(1 - \frac{2m}{r})\mathbf{dt}^2 + \frac{\mathbf{dr}^2}{(1 - \frac{2m}{r})} + r^2(\mathbf{d}\theta^2 + \sin^2\theta \mathbf{d}\phi^2))$$

The more general case, (as seen for instance in [Hawk], appendix B) for a spherical spacetime is,

$$\mathbf{ds}^2 = -\frac{\mathbf{dt}^2}{F^2(r)} + X^2(r)\mathbf{dr}^2 + r^2(\mathbf{d}\theta^2 + \sin^2\theta \mathbf{d}\phi^2)) \tag{2.1}$$

We thus have two positive radial functions (assumed to be at least C^2) determining the metric: for consistency with [FSY1], we will write the metric tensor in the form

$$g_{ij} = diag(\frac{1}{T^2}, -\frac{1}{A}, -r^2, -r^2\sin^2\theta) \tag{2.2}$$

with A and T positive functions of r. The Dirac operator must now be constructed in these coordinates, to take advantage of this symmetry.

Since the Dirac matrices are elements of a spinor space, and correspond by the homomorphism $\mathcal{H}$ (defined in Appendix A) with elements of the tangent space, they transform in the same way. Thus, since the $\mathbf{e}_j = \frac{\partial}{\partial x^j}$, the Dirac matrices transform the same way, and so we have:

$$\begin{aligned}
G^t &= T\gamma_0 \\
G^r &= \sqrt{A}(\gamma_1\cos\theta + \gamma_2\sin\theta\cos\phi + \gamma_3\sin\theta\sin\phi) \\
G^\theta &= \frac{1}{r}(-\gamma_1\sin\theta + \gamma_2\cos\theta\cos\phi + \gamma_3\cos\theta\sin\phi) \\
G^\phi &= \frac{1}{r^2\sin\theta}(-\gamma_2\sin\phi + \gamma_3\cos\phi)
\end{aligned} \tag{2.3}$$

and in particular, we have (as a representative example):

$$G^\phi = \begin{pmatrix} 0 & 0 & \frac{\cos(\phi)}{r} & \frac{i\sin(\phi)}{r} \\ 0 & 0 & -\frac{i\sin(\phi)}{r} & -\frac{\cos(\phi)}{r} \\ -\frac{\cos(\phi)}{r} & -\frac{i\sin(\phi)}{r} & 0 & 0 \\ \frac{i\sin(\phi)}{r} & \frac{\cos(\phi)}{r} & 0 & 0 \end{pmatrix}$$

with the other G^k found similarly. Now we recall that $\rho = \frac{i}{4!}\epsilon_{ijkl}G^iG^jG^kG^l$, and we may calculate the form of ρ in these coordinates by using the forms for the G^k given above. Since we know that the G^k anti-commute when indices are different, and ϵ_{ijkl} is zero when any index is repeated, so that $\rho = \frac{i}{\sqrt{|g|}}G^tG^rG^\theta G^\phi$, which an explicit calculation reveals to be the "pseudoscalar" matrix

$$\rho = \gamma_5 \cong i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

which is convenient, since it is independent of position. We recall the form of (A.3), the components of the spin derivative. With ρ a constant, the term $\frac{i}{2}\rho(\partial_j\rho)$ can be eliminated. With the particular value of ρ we have found, the term $\frac{i}{8}Tr(\rho G_j\nabla_m G^m)\rho$ can also be found by direct calculation to be zero. This leaves the B matrices as:

$$B(x) = G^j(x)E_j(x) = -\frac{i}{16}Tr(G^m\nabla_j G^n)G^jG_mG_n$$

Consider the term $G^jG_mG_n$: since $G^jG^k = g^{jk}$, we get nonzero terms for every pair of equal indices (with possible sign changes due to the anti-commutativity), while the relation between the G matrices and $\rho \cong \gamma^5$, a self-inverse matrix, gives us a remaining nonzero term of $(i\epsilon^j_{mnp}\gamma^5G^p)$, so that we have the following:

$$B = -\frac{i}{16}Tr(G^m\nabla_j G^n)(\delta^j_m G_n - \delta^j_n G_m + G^j g_{mn} + i\epsilon^j_{mnp}\gamma^5 G^p$$

With a few more observations, we can simplify this greatly. First, we note that since $4g^{mn}$ is the trace of the matrix G^mG^n, and $g^{mn}{}_{;j}$ vanishes, we have $0 = \nabla_j Tr(G^mG^n)$ which is just $Tr((\nabla_j G^m)G^n) + Tr(G^m(\nabla_j G^n))$, so that the first two terms in the last expression for B are equal, while the third is zero (since it corresponds to the case with equal values for m and n). Furthermore, in the term involving the volume tensor ϵ, the antisymmetry of this tensor allows us to replace the covariant derivative in the multiplying term with a partial derivative, and this in turn means that the trace in the multiplying term is zero in any term with all different tensor indices (i.e. the terms with nonzero ϵ). This last is a somewhat cumbersome calculation, which can be checked with a symbolic computation program, using the explicit forms of the G. This means that only the first two terms (which are equal, due to the antisymmetry) are significant, and so we have

$$B = \frac{i}{8} Tr(G^n \nabla_j G^j) G_n$$

This can be somewhat further simplified by nothing that, as with basis vectors, we have $\nabla_j G^j$ to be a linear combination of the G matrices themselves, and that $Tr(G^n \mathbf{X}) G_n = 4\mathbf{X}$ for any such combination, so that finally we have the quite simple form:

$$B = \frac{i}{2} \nabla_j G^j$$

Having found this convenient form for the B matrices, we can find it explicitly, and thus obtain a form for the Dirac operator with which to do calculations. To find the divergence $\nabla_j G^j$, we note that it is just $\frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|} G^j)$. Thus, we must find the derivatives of the G matrices found earlier. Noting that we chose the form of the matrices to reflect a static geometry, the term for the t index is naturally zero. For the radial component, we find:

$$\begin{aligned}\frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}G^r) =&\Big(\frac{TA^{\frac{1}{2}}}{r^2\sin(\theta)}\Big)\frac{\partial}{\partial r}\Big(\big(\frac{r^2\sin(\theta)}{T\sqrt{A}}\big)G^r\Big)\\ =&\Big(\frac{TA^{\frac{1}{2}}}{r^2}\Big)\frac{\partial}{\partial r}\Big(\big(\frac{r^2}{T}\big)((\gamma^1\cos(\theta)\\ &+\gamma^2\sin(\theta)\cos(\phi)+\gamma^3\sin(\theta)\sin(\phi)))\Big)\end{aligned}$$

Here we have removed some factors of $\sqrt{A}$ and non-radially-dependent parts of the expression. Noting that the γ^i are not radially dependent, this leaves only:

$$\Big(\frac{2}{r}-\frac{\partial_r T}{T}\Big)G^r$$

By similar calculations, we find that:

$$\begin{aligned}\frac{1}{\sqrt{|g|}}\partial_\theta(\sqrt{|g|}G^\theta) =&\frac{1}{r\sin(\theta)}\Big(-2\gamma^1\sin(\theta)\cos(\theta)\\ &+(\cos^2(\theta)-\sin^2(\theta))(\gamma^2\cos(\phi)+\gamma^3\sin(\phi))\Big)\end{aligned}$$

and

$$\frac{1}{\sqrt{|g|}}\partial_\phi(\sqrt{|g|}G^\phi) = \frac{1}{r\sin(\theta)}(-\gamma^2\cos(\phi)-\gamma^3\sin(\phi))$$

Summing these to obtain the divergence $\nabla_j G^j$, we get that the B matrices are given by:

$$B = \frac{i}{2}(\frac{2}{r}-\frac{\partial_r T}{T})G^r - \frac{i}{r}(\gamma^1\cos(\theta)+\gamma^2\sin(\theta)\cos(\phi)+\gamma^3\sin(\theta)\sin(\phi))$$

and noting that this last combination is a scalar multiple of G^r, we can reduce this finally to:

$$B = i\left(\frac{1}{r}(1-\frac{1}{\sqrt{A}})-\frac{1}{2}\frac{\partial_r T}{T}\right)G^r$$

Finally, we combine this with the form for the Dirac operator and get it to be:

$$G = iG^t \frac{\partial}{\partial_t} + iG^r \left(\frac{\partial}{\partial r} + \frac{1}{r}(1 - \frac{1}{\sqrt{A}}) - \frac{1}{2}\frac{\partial_r T}{T} \right) \tag{2.4}$$

This is perhaps the clearest form in which the operator can be written. In the next section, we shall examine it, and reduce the expression further by exploiting further symmetries. Along the way, we shall give some discussion about some more general considerations.

2.1.2 Refinements

We have developed a form for the Dirac operator: before proceeding with further we must check that this operator is Hermitian with respect to the appropriate scalar product, since physical observables in quantum mechanical systems correspond to Hermitian operators. We shall now provide a short discussion of this in the current context, more details on which may be found in [Fin].

There are two scalar products defined for solutions to the Dirac equation. The first of these applies to any wave function: integrating, over all of spacetime, the scalar product of two functions with the invariant measure accounting for the tensor density:

$$< \Psi, \Phi >= \int_{\mathcal{M}} \overline{\Psi} \Phi \sqrt{|g|} d^4 x \tag{2.5}$$

In this case, the bar represents the adjoint operation on spinors, so that

$$\overline{\Phi} = \Phi^* \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

This scalar product is called the *spin scalar product*, and is indefinite of signature $(2,2)$. In order to give meaning to the scalar product as a probability density, we would like to have a scalar product, however, which does not involve integration over all of spacetime, but rather only on some spacelike hypersurface. This is generally written as

$$(\Psi|\Phi) = \int_{\mathcal{H}} \overline{\Psi} G^j \Phi \nu_j d\mu \tag{2.6}$$

where $\mathcal{H}$ is any such hypersurface, on which $d\mu$ is the measure induced by the metric. This scalar product has a physical interpretation as the probability density of the Dirac particle, and the conservation of the Dirac current means that it is independent of the choice of $\mathcal{H}$ - that is, $\mathcal{H}$ can be continuously deformed to any other spacelike hypersurface and leave $(\Psi|\Phi)$ fixed. Conservation of the Dirac current is the statement that $\nabla_j \overline{\Psi} G^j \Phi = 0$, which holds for solutions to the Dirac equation. The spin scalar product described above is more general, since it makes the time coordinate into an observable, but lacks an immediate physical interpretation - the second will be used to give the normalization conditions which we shall use later, since it has physical meaning.

We must check that the Dirac operator is Hermitian with respect to the spin scalar product, so that it will correspond to an observable quantity. (The fact that the operator G is Hermitian with respect to the spin scalar product justifies the notation, commonly used in quantum field theory, $< \Psi|G|\Phi >$, the bra/ket notation.) This is easy to check, using our explicit form for the

Dirac operator with the B matrices in the form of a divergence:

$$\begin{aligned}
< G\Psi|\Phi > &= \int \overline{\Big(iG^j\frac{\partial}{\partial x^j}+\frac{i}{2}\nabla_j G^j\Big)\Psi}\Phi\sqrt{|g|}d^4x \\
&= \int \overline{\Psi}\Big(iG^j\frac{\partial}{\partial x^j}-\frac{i}{2}\nabla_j G^j\Big)\Phi\sqrt{|g|}d^4x \\
&\quad + \int \overline{\Psi}\Big(i\partial_j(\sqrt{|g|}G^j)\Big)\Phi d^4x \\
&= \int \overline{\Psi}\Big(iG^j\frac{\partial}{\partial x^j}+\frac{i}{2}\nabla_j G^j\Big)\Phi\sqrt{|g|}d^4x \\
&= < \Psi|G\Phi >
\end{aligned} \tag{2.7}$$

This provides some justification for the physical significance of the operator G which we have constructed. Note that in this calculation, we have simply moved the term $\int\overline{\Psi}\Big(i\partial_j(\sqrt{|g|}G^j)\Big)\Phi d^4x$ from the contribution of the first, flat-space G, term in the integral to the second, B, term and thus removing the conjugation.

We have checked the meaningfulness of the Dirac operator in the most recently obtained form (2.4) by checking that it is Hermitian with respect to the spin scalar product. To simplify calculations, we shall next separate out the angular momentum from the equation, in order to simplify it. This will involve the use of an ansatz for the wave functions: here, we are simply assuming that the wave functions can be expressed in a particular form, and use the characteristics of that form to simplify the equation. While this is far from obvious a-priori, we will attempt to justify the use of the ansatz.

First, we make some definitions. By analogy with the construction of the Dirac spinors from the Pauli spinors in flat spacetime, we define some combinations of Pauli matrices which correspond to our new coordinate system. These will capture dependence on the angle coordinates. In particular, define (with the usual definitions (A.1) for the σ^i matrices in cartesian coordinates):

$$\begin{aligned}
\sigma^r(\theta,\phi) &= \cos(\theta)\sigma^1 + \sin(\theta)\cos(\phi)\sigma^2 + \sin(\theta)\sin(\phi)\sigma^3 \\
\sigma^\theta(\theta,\phi) &= -\sin(\theta)\sigma^1 + \cos(\theta)\cos(\phi)\sigma^2 + \cos(\theta)\sin(\phi)\sigma^3 \\
\sigma^\phi(\theta,\phi) &= \frac{1}{\sin(\theta)}(-\sin(\phi)\sigma^2 + \cos(\phi)\sigma^3)
\end{aligned} \tag{2.8}$$

We now seek a convenient form for the wave functions which allows us to simplify our system (2.4). Specifically, we assume that the wave function takes the form:

$$\Phi_a = e^{i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a \end{pmatrix} \tag{2.9}$$

where the u_i are complex radial functions and the e_a are the standard basis of the Pauli spinors, namely $e_1 = (1,0)$ and $e_2 = (0,1)$. The new term ω is a parameter which we shall end up using to classify solutions: it represents the energy of the system. This form will be seen to be quite convenient, and rather general, as we shall now attempt to show. In general, we can, with the same definition of the e_a, express a totally general form for the Dirac wave functions for two Dirac particles as $\Phi_a(\mathbf{x},t) = A(\mathbf{x},t)e_a$. This A is a (4×2) matrix whose columns are then the components of the wave functions for the two Dirac particles - it represents the combined system. We want the evolution of this system to produce a static solution, and for this system to be static, we must have the evolution of A to be only a change in phase:

$$A(\mathbf{x},t) = -e^{i\omega t}A(\mathbf{x})$$

We see that the two Dirac particles then have an oscillation with a frequency proportional to the energy ω. On the other hand, if we want to have spherical symmetry, we must have no angular dependence: in terms of our Pauli-matrix expressions (2.8), we must have A in terms of only the t and r spinors given in our new system of σ matrices. These are $\mathbb{I}$ and σ^r as given previously. Thus, we have:

$$A(\mathbf{x}) = \begin{pmatrix} v_1(r)\mathbb{I} + v_2(r)\sigma^r \\ v_3(r)\mathbb{I} + v_4(r)\sigma^r \end{pmatrix}$$

This form for the combined system of two Dirac particles, when we multiply this matrix form by the e_a, gives a linear combination of the form (2.9) given above for the wave function, and its counterpart with the σ^r spinor in the top entry instead of the bottom, namely:

$$\Phi_a = e^{i\omega t} \begin{pmatrix} \sigma^r u_1 e_a \\ u_2 e_a \end{pmatrix} \tag{2.10}$$

This last variant ansatz for the solution will produce a solution to a transformed version of the Dirac equation in which the mass is negative, corresponding to conjugation by γ^5 of the Dirac operator. We consider the combination of both of these cases, corresponding to the fact that the Dirac equation has meaningful solutions of both positive and negative mass, a fact which leads to the "Dirac sea" of negative-mass solutions. We shall regard these solutions as transformed versions of solutions of the untransformed Dirac equation, and so consider them in our analysis of the solutions, when we find them. This means that we can consider only one of the two forms as the ansatz for the metric (breaking the symmetry of positive and negative mass by choosing one to work with), and obtain (2.9) as the ansatz we shall use.

With this in mind, we proceed to use the form (2.9) to separate the angular momentum from the Dirac equation. To do this, we find the form of the Dirac operator acting on such a wave function. We have, first, that:

$$G\Psi_a = \big(iG^t\partial_t + G^r(i\partial_r + \frac{i}{r}(1-A^{-\frac{1}{2}}) - \frac{i}{2}\frac{T'}{T}) + iG^\theta\partial_\theta + iG^\phi\partial_\phi\big)\Psi_a$$

This is just:

$$G\Psi_a = \Big[iG^t\partial_t + iG^r(\partial_r + \frac{1}{r}(1-A^{-\frac{1}{2}}) - \frac{T'}{2T})\Big]e^{i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a\end{pmatrix}$$

Now, the first term (involving the ∂_t derivatives) becomes:

$$\begin{aligned}(iG^t\partial_t)\Big(e^{-i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a\end{pmatrix}\Big) &= (iG^t)(-i\omega e^{-i\omega t})\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a\end{pmatrix} \\ &= G^t\omega\Psi_a \\ &= \omega T\gamma^0\Psi_a\end{aligned}$$

The term involving ∂_r derivatives (and the corresponding spin-derivative corrections) becomes:

$$iG^r\Big(\partial_r + \frac{1-A^{-\frac{1}{2}}}{r} - \frac{T'}{2T}\Big)\Psi_a = \begin{pmatrix} 0 & \sigma_r \\ -\sigma^r & 0\end{pmatrix}\Big(i\partial_r + i\frac{\sqrt{A}-1}{r} - i\frac{T'\sqrt{A}}{2T}\Big)\Psi_a$$

Now, since the angular derivatives are zero for radial functions, the angular derivative terms' only effect comes from their action on σ^r, where we have that $\sigma^r(\partial_\theta\sigma^r) = \mathbb{I}$, and similarly for the ϕ derivative. Thus, we have:

$$\begin{aligned}(iG^\theta\partial_\theta)\left(e^{-i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a\end{pmatrix}\right) &= (iG^\theta)(e^{-i\omega t})\Big(\partial_\theta\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a\end{pmatrix}\Big) \\ &= (iG^\theta)(e^{-i\omega t})\begin{pmatrix} \partial_\theta(u_1 e_a) \\ \partial_\theta(\sigma^r u_2 e_a)\end{pmatrix} \\ &= (iG^\theta)(e^{-i\omega t})\begin{pmatrix} 0 \\ (\partial_\theta\sigma^r)(u_2 e_a)\end{pmatrix}\end{aligned}$$

And since, furthermore, $G^\theta = \frac{1}{r}\begin{pmatrix} 0 & \sigma^\theta \\ -\sigma^\theta & 0 \end{pmatrix}$, this simply becomes:

$$\begin{aligned}
(iG^\theta \partial_\theta)\left(e^{-i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a \end{pmatrix}\right) &= \frac{i}{r}(e^{-i\omega t})\begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix}\begin{pmatrix} 0 \\ (\partial_\theta \sigma^r)(u_2 e_a) \end{pmatrix} \\
&= \frac{i}{r}(e^{-i\omega t})\begin{pmatrix} 0 \\ u_2 e_a \end{pmatrix} \\
&= \frac{i}{r}\begin{pmatrix} 0 & \sigma^r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a \end{pmatrix}
\end{aligned}$$

since $(\sigma^r)^2 = \mathbb{I}$. All of this applies equally to the ∂_ϕ term, so these two terms are equal, and the whole expression becomes:

$$\begin{aligned}
G\Psi_a =& \left(\begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix} i\left(\sqrt{A}\partial_r + \frac{\sqrt{A}-1}{r} - \frac{\sqrt{A}T'}{2T}\right)\right. \\
&\left. + \omega T\gamma^0 + \frac{2i}{r}\begin{pmatrix} 0 & \sigma^r \\ 0 & 0 \end{pmatrix}\right)\Psi_a
\end{aligned} \tag{2.11}$$

Notice that this form produces a coupled system of differential equations in the two unknown functions u_1 and u_2 which appear in the ansatz for Ψ. This equation can now be simplified by removing the angular momentum: we accomplish this by first rewriting this equation in a more convenient form by transforming Ψ. The form we shall choose makes it easy to write the Dirac equation as an ordinary differential equation (or rather a system of them), involving only radial derivatives. This will be done by solving for different functions of r, T and the u_a, from which they can be recovered. In particular, we will consider functions which simplify the form of the above equation, rendering it real rather than fully complex, and eliminating some of its terms, namely:

$$\Phi_1 = \frac{r}{\sqrt{T}} u_1$$

and

$$\Phi_2 = -\frac{ir}{\sqrt{T}} u_2$$

With this substitution, we consider the expression

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \omega T - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{A}\partial_r + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{1}{r} - m \right) \Phi = 0 \tag{2.12}$$

For u for which the Dirac equation $(G - m)\Psi_a = 0$ hold, this last equations holds also, and vice versa, since they are scalar multiples of each other. Since this equation is real, we may assume that the spinor Φ is real.

We note that the normalization condition now becomes simply

$$\int_0^\infty |\Phi|^2 \frac{T}{\sqrt{A}} dr = \frac{1}{4\pi}$$

To give the simplest form for the Dirac equation as an ODE, we choose the form

$$\sqrt{A}\Phi' = \left[\omega T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \Phi \tag{2.13}$$

which we shall use for the numerical computations.

2.2 The Field Equations

2.2.1 The Energy-Momentum Tensor

We have found the Dirac equations already: we wish now to find the Einstein field equations, so that we may attempt to find solutions for the coupled system. The standard way to do this is by variational methods (see e.g. [LoRu] chapter 8.4). The idea here is that S is a Lagrangian dependent on the metric and its first two derivatives, whose associated energy-momentum tensor has components which are its derivatives with respect to the metric components. We want to find this tensor associated to the Dirac field in order to determine its effect on the curvature of the metric, and hence on the metric itself. This produces the coupling of the matter field to the metric used in the Einstein equations, and will thus give us the next equations we need. To do this we must find the variation of the metric in order to compute the derivatives.

If we allow ourselves to consider a variation of the metric which is arbitrary, say δg_{ij}, then we can use algebraic relations to discover the variation of other quantities. In particular, the variation of the Dirac matrices corresponding to a given δg_{ij} arises in the following way. Since we have the relation $\frac{1}{2}(G^i G^j + G^j G^i) = g^{ij}$, and the variation of the inverse is $\delta g^{ij} = -g^{ik} g^{jl} \delta g_{kl}$, we can differentiate these to find the variation for the covariant and contravariant Dirac spinors to be:

$$\delta G_j = \frac{1}{2}(\delta g_{jk}) G^k$$

and

$$\delta G^j = -\frac{1}{2} g^{jk} (\delta g_{kl}) G^l$$

We can use these basic quantities to find the energy-momentum tensor: this can be found as the variation of the action of the Dirac operator, which is just:

$$S = \int \overline{\Psi}(G - m)\Psi \sqrt{|g|} d^4x$$

So the variation is δS, or, since Ψ solves the Dirac equation, and hence $(G - m)\Psi = 0$, we can find δS by considering only the contribution of δG, so that we get:

$$\delta S = \int Re\hat{\Psi}\big(i(\delta G^j)\frac{\partial}{\partial x^j} + \delta B\big)\Psi\sqrt{|g|}d^4x$$

Note that since the action is real, we have removed the imaginary part of the integrand. Next, we simplify this form by showing that the contribution of the B matrices to the variation of the action is in fact zero. We begin by allowing the variation δB to act on Ψ, substituting the form for B which we found previously:

$$\begin{aligned} B &= G^j E_j \\ &= G^j\Big(\frac{i}{2}\rho(\partial_j\rho) - \frac{i}{16}Tr(G^m\nabla_j G^n)G_m G_n + \frac{i}{8}Tr(\rho G_j \nabla_m G^m)\rho\Big) \end{aligned}$$

Noting that the first term is zero since ρ is constant, and the third term is traceless, we find the contribution of the B matrices to the integrand in the variation to be:

$$\begin{aligned} Re\overline{\Psi}\delta B\Psi &= \frac{1}{16}Im\delta\Big(Tr(G^m\nabla_j G^n)\overline{\Psi}G^j G_m G_n\Psi\Big) \\ &= \frac{1}{16}\delta\Big(Tr(G^m\nabla_j G^n)Im(\overline{\Psi}G^j G_m G_n\Psi)\Big) \end{aligned}$$

Here, we notice that since $\gamma^5 \cong \rho = \epsilon_{ijkl}G^i G^j G^k G^l$, we can replace the term $G^j G_m G_n$, and then use the antisymmetry of the tensor density ϵ to convert covariant to partial derivatives, and get that:

$$\begin{aligned} Re\overline{\Psi}\delta B\Psi &= \frac{1}{16}\delta\Big(\epsilon^{jmnp}Tr(G_m\nabla_j G^n)\overline{\Psi}\gamma^5 G_p\Psi\Big) \\ &= \frac{1}{16}\delta\Big(\epsilon^{jmnp}Tr(G_m\partial_j G^n)\overline{\Psi}\gamma^5 G_p\Psi\Big) \\ &= \frac{1}{16}\Big(\epsilon^{jmnp}\delta Tr(G_m\partial_j G^n)\overline{\Psi}\gamma^5 G_p\Psi\Big) \end{aligned}$$

where the last transformation is a result of the antisymmetry of ϵ and the fact that $Tr(G_m\partial_j G_n) = 0$ when there are no repeated indices, so that we can ignore the ϵ^{jmnp}'s contribution to the variation. Substituting for the variation in the middle, we then get that

$$Re\overline{\Psi}\delta B\Psi = \frac{1}{16}\Big(\epsilon^{jmnp}(\delta G_{mk})Tr(G^k\partial_j G^n)\overline{\Psi}\gamma^5 G_p\Psi\Big)$$

To reduce this further, we observe that the last peice,

$$\overline{\Psi}\gamma^5 G_p\Psi = \sum_{a=1}^{2}\overline{\Psi_a}\gamma^5 G_p\Psi_a = 0$$

and so the variation of the B matrices disappears. Thus, we need only find the variation δG^j to find δS. This is thus:

$$\delta S = \int \frac{1}{2}\sum_{a=1}^{2} Re\overline{\Psi_a}(iG_j\partial_k)\Psi_a\delta g^{jk}\sqrt{|g|}d^4x$$

and the energy-momentum tensor is the symmetrized form of this:

$$T_{jk} = \frac{1}{2}\sum_{a=1}^{2} Re\overline{\Psi_a}(iG_j\partial_k + iG^k\partial_j)\Psi_a$$

Now it is easy to show from the algebraic properties of the Pauli matrices that the cross terms with $j \neq k$ vanish, and direct calculation gives the others as:

$$T^t_t = 2\omega T^2 r^{-2}|\Phi|^2 \tag{2.14}$$

$$T^r_r = -2\omega T^2 r^{-2}|\Phi|^2 + 4Tr^{-3}\Phi_1\Phi_2 + 2mTr^{-2}(\Phi_1^2 - \Phi_2^2) \tag{2.15}$$

$$T^\theta_\theta = -2r^{-3}T\Phi_1\Phi_2 \tag{2.16}$$

$$T^\phi_\phi = -2r^{-3}T\Phi_1\Phi_2 \tag{2.17}$$

2.2.2 Field Equations

Now that we have obtained the Energy-Momentum tensor for the Dirac field, we must find the Einstein Field Equations which are obtained from it. We recall the form of the metric given in (2.2); this metric is the most general spherically symmetric metric possible, and has two arbitrary functions, namely A and T, of the radius coordinate r. We thus wish to find the components of the Einstein tensor $G^i_j = R^i_j - \frac{1}{2}R\delta^i_j$ in terms of $A(r), T(r)$ and their radial derivatives. The part of the field equations which derive from the metric will be this tensor, hence these components.

Calculating the Einstein tensor for the given metric, we find that it is a diagonal matrix with the following components (all primes represent radial derivatives):

$$\begin{aligned} G^0_0 &= -\frac{1}{r^2} + \frac{A}{r^2} + \frac{A'}{r} \\ G^1_1 &= -\frac{1}{r^2} + \frac{A}{r^2} - \frac{2AT'}{rT} \\ G^2_2 = G^3_3 &= \frac{A'}{2r} - \frac{AT'}{rT} - \frac{A'T'}{2T} + \frac{2AT'^2}{T^2} - \frac{AT''}{T} \end{aligned} \tag{2.18}$$

Now, the Einstein field equations $G^i_j = -8\pi T^i_j$, using the form for the energy-momentum tensor T^i_j calculated in (2.14) give three equations, since T^i_j is again a diagonal matrix with the last two entries equal. The first of these equations is:

$$\begin{aligned} G^0_0 &= -8\pi T^0_0 \\ -\frac{1}{r^2} + \frac{A}{r^2} + \frac{A'}{r} &= -8\pi \frac{2\omega T^2|\Phi|^2}{r^2} \\ -(1-A) + rA' &= -16\pi\omega T^2|\Phi|^2 \end{aligned} \tag{2.19}$$

the second is:

$$\begin{aligned} G^1_1 &= -8\pi T^1_1 \\ -\frac{1}{r^2} + \frac{A}{r^2} - \frac{2AT'}{rT} &= -8\pi \frac{-2\omega T^2|\Phi|^2 + 4Tr^{-1}\Phi_1\Phi_2 + 2mT(\Phi_1^2 - \Phi_2^2)}{r^2} \\ (1-A) + \frac{2rAT'}{T} &= -16\pi\omega T^2|\Phi|^2 + 32\pi Tr^{-1}\Phi_1\Phi_2 + 16\pi mT(\Phi_1^2 - \Phi_2^2) \end{aligned} \tag{2.20}$$

and the last field equation is:

$$G^3_3 = G^2_2 = -8\pi T^2_2 = -8\pi T^3_3$$

$$\frac{A'}{2r} - \frac{AT'}{rT} - \frac{A'T'}{2T} + \frac{2AT'^2}{T^2} - \frac{AT''}{T} = -8\pi\frac{-2Tr^{-1}\Phi_1\Phi_2}{r^2}r^2 \tag{2.21}$$

$$\frac{rA'}{2} - \frac{rAT'}{T} - \frac{r^2A'T'}{2T} + \frac{2r^2AT'^2}{T^2} - \frac{r^2AT''}{T} = -16\pi Tr^{-1}\Phi_1\Phi_2$$

Together, these form the Einstein part of the Einstein-Dirac system of equations for the case we are considering. We can combine these with the equation we derived (2.13) for the Dirac equation in the spherically symmetric background, namely:

$$\sqrt{A}\Phi' = \left(\omega T\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \frac{1}{r}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - m\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\Phi$$

Together, these comprise the system of equations in whose solutions we are interested. In fact, the system can be simplified by showing that the equation (2.21) can be eliminated from this system, as it is implied by the others: we shall show this once we have the equations in a convenient form.

Having eliminated the equation (2.21) from our system, we wish to find a consistent form for our system to make it easier to work with. To do this, we will first isolate single terms with unknown functions so that we may sequentially solve for these functions. It is easy to see that if we take the Dirac equations (2.13) from matrix form, and writing $\alpha = \Phi_1$ and $\beta = \Phi_2$ they appear as:

$$\sqrt{A}\alpha' = \frac{1}{r}\alpha - (\omega T + m)\beta \tag{2.22}$$

and

$$\sqrt{A}\beta' = (\omega T - m)\alpha - \frac{1}{r}\beta \tag{2.23}$$

With the same conventions for α and β, the remaining Einstein equations, on the other hand, may be written (by isolating terms with derivatives) as:

$$rA' = 1 - A - 16\pi\omega T^2(\alpha^2 + \beta^2) \tag{2.24}$$

and

$$2rA\frac{T'}{T} = A - 1 - 16\pi\omega T^2(\alpha^2 + \beta^2) + \frac{32\pi T\alpha\beta}{r} + 16\pi mT(\alpha^2 - \beta^2) \tag{2.25}$$

With these four equations, we have reduced the problem to the solution of solving for a system of four unknown functions satisfying these relations in radial derivatives.

Now recall the normalization condition on the wave function: since the wave function may be considered as a point in a projective Hilbert space, only values which are scaled to have magnitude 1 are physically meaningful (this represents the probability amplitude of the system). The magnitude is defined by the scalar product given previously in (2.6), namely

$$(\Psi|\Phi) = \int_{\mathcal{H}} \overline{\Psi} G^j \Phi \nu_j d\mu$$

with $\mathcal{H}$ a spacelike hypersurface and $d\mu$ the measure induced on it by the metric. Requiring that Φ have magnitude 1 in the norm induced by this scalar product on solutions to the Dirac equation leads to physically meaningful solutions. Taking $\mathcal{H}$ to be surfaces of constant time parameter t, we can integrate this radially, getting the scaling condition to then be

$$\int_0^\infty |\Phi|^2 \frac{T}{\sqrt{A}} dr = \frac{1}{4\pi} \tag{2.26}$$

since we just have the normal vector ν_j with only a time component, hence picking out only the G^0 term.

This normalization condition, together with the assumption that the solution is regular at $r = 0$ (that is, can be expanded as a Taylor series), leads us to the Taylor series expansions, to second order, for the four functions $\alpha, \beta, A \text{and} T$ of r given here:

$$\alpha(r) = \alpha_1 r + \mathcal{O}(r^3) \tag{2.27}$$

(which is nearly tautological, except that it reveals $\alpha_0 = 0$),

$$\beta(r) = \frac{1}{3}(\omega T_0 - m)\alpha_1 r^2 + \mathcal{O}(r^3) \tag{2.28}$$

$$A(r) = 1 - \frac{2}{3}\omega T_0^2 \alpha_1^2 r^2 + \mathcal{O}(r^3) \tag{2.29}$$

and

$$T(r) = T_0 - \frac{m}{6}(4\omega T_0 - 3m)T_0^2 \alpha_1^2 r^2 + \mathcal{O}(r^3) \tag{2.30}$$

In these expressions, we recall that ω and m are the energy and (rest) mass of the Dirac particle, respectively, and are thus preexisting parameters for the system. This Taylor expansion shows us that, in this form, the degrees of freedom for the solutions are then determined by two additional parameters: the value $T_0 = T(0)$ and the value $\alpha_1 = \frac{d\alpha}{dr}$. We can restrict this further by noting some additional restrictions on the form of the solutions.

First, we recall that the ADM (Arnowitt-Deser-Misner) mass of a system is a concept of the total mass of a system, originally motivated by the Hamiltonian formulation of General Relativity, in which the existence of constraints in the formulation lead one to seek to "de-parameterize" the theory, leading to a precise notion of total energy in a system (the Hamiltonian).

This led to the ADM mass, which we may think of as the mass of a system as measured by an observer at spatial infinity. Requiring that this be finite leads to the constraint:

$$\lim_{r \mapsto \infty} \frac{r}{2}(1 - A(r)) < \infty \tag{2.31}$$

but since $\lim_{r \mapsto \infty} r = \infty$, this implies that

$$\lim_{r \mapsto \infty} A(r) = 1 \tag{2.32}$$

This constraint eliminates one degree of freedom for our system, corresponding to one parameter.

The second constraint (eliminating the second spurious parameter form our Taylor series formulation) is simply that we wish our spacetime to be asymptotically flat - that is, asymptotically Minkowskian. Given the previous constraint and our form of the metric in (2.2), this leads to the remaining constraint

$$\lim_{r \mapsto \infty} T(r) = 1 \tag{2.33}$$

We have now nearly obtained a form for our Einstein-Dirac equations which will be susceptible of numerical treatment. The normalization condition and the asymptotic flatness condition ((2.26) and (2.33)) are difficult to make use of, however, in a numerical context. Instead, we shall make use of a re-parameterization technique which will make it possible to substitute the integral normalization condition and the asymptotic flatness conditions with finiteness conditions and explicit choice of values for some parameters, identifying solutions of the equations thus discovered with solutions of the desired Einstein-Dirac equations under a scaling of the coordinates about the point $r = 0$.

In particular, we shall replace the abovementioned constraints with the finiteness conditions:

$$\lim_{r \mapsto \infty} T(r) < \infty \tag{2.34}$$

and

$$\int_0^\infty |\Phi|^2 \frac{T}{\sqrt{A}} dr < \infty \tag{2.35}$$

while compensating for the extra degrees of freedom gained by setting

$$T_0 = 1 \tag{2.36}$$

and

$$m = \pm 1 \tag{2.37}$$

(Note that $m = \pm 1$ includes both the positive and negative mass solutions for the Dirac equation, and that these are treated separately. The negative-mass solutions are those discovered by Dirac, forming the "sea", holes in which are detected as antimatter Dirac particles.)

The coordinate transformation which makes these constraints equivalent to the first set involves first a scaling of r by a factor of

$$\lambda = \sqrt{4\pi \int_0^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} dr}$$

(which is the of the ratio of the actual norm of the wave function with the desired value of 1). Then, if we take

$$\tau = \lim_{r \mapsto \infty} T(r)$$

we can take a solution (α, β, T, A) of the equations (2.22), (2.23), (2.24) and (2.25) satisfying our new constraints (2.34) and (2.35), then we can produce one satisfying the old constraints (2.26) and (2.33) by defining the new functions

$$\begin{aligned}
\tilde{\alpha} &= \sqrt{\frac{\tau}{\lambda}}\alpha(\lambda r) \\
\tilde{\beta} &= \sqrt{\frac{\tau}{\lambda}}\beta(\lambda r) \\
\tilde{A} &= A(\lambda r) \\
\tilde{T} &= \tau^{-1}T(\lambda r)
\end{aligned} \tag{2.38}$$

It is clear that, for strictly positive λ,τ, this transformation is invertible. Further, this new wave function satisfies the Einstein-Dirac equations (2.22), (2.23), (2.24) and (2.25) with the parameters

$$\begin{aligned}
\tilde{m} &= \lambda m \\
\tilde{\omega} &= \lambda \tau \omega
\end{aligned} \tag{2.39}$$

as can be checked by direct substitution. Also, it is clear that these functions satisfy the conditions

$$\begin{aligned}
\int_0^\infty (\tilde{\alpha}^2 + \tilde{\beta}^2)\frac{\tilde{T}}{\sqrt{\tilde{A}}}dr &= \frac{1}{4\pi} \\
\lim_{r \mapsto \infty} \tilde{T}(r) &= 1 \\
\lim_{r \mapsto \infty} \frac{r}{2}(1 - \tilde{A}(r)) &< \infty
\end{aligned} \tag{2.40}$$

as required. Thus, these provide us a unique solution for our Einstein-Dirac equations, corresponding to the solution we found for the numerically tractable system (2.22), (2.23), (2.24), (2.25) with constraints (2.34) and (2.35). Note that since the normalization condition and the asymptotic flatness condition are required for physical significance, the scaled solutions are the ones in whose properties we shall be interested.

Chapter 3

Solutions to the Dirac Equation and their Properties

Having found Taylor expansions for the relevant physical quantities α, β, A and T about the origin (giving initial conditions) and the ODEs which they satisfy, one may proceed to solve these ODEs numerically. This has been done (see [FSY1] sections 7, 8) but we do not propose to present extensive details here on the nature of these solutions. These details may be found in [FSY1] if necessary - in particular, the graphs of the various functions α, β, A and T being sought are of some interest. Our main purpose here, however, is to describe the method for finding these solutions, and a few of their most salient qualitative properties. We will then proceed to examine the stability of these solutions under perturbation, which will give some idea as to whether these states represent physically realistic situations. This will involve some topological properties of the solutions.

3.1 Existence

3.1.1 Determination and Properties of Solutions

We wish here to consider how one would go about finding useful solutions to the differential equations (2.22)-(2.25). This has been done by Finster, Smoller and Yau ([FSY1]). In order to find numerical solutions for the system under consideration, one would use the Taylor expansions we have obtained already in (2.27)-(2.30) to construct initial data about the origin, and then use a numerical DE solver to use these initial conditions to develop a full solution (the Taylor expansions are necessary so that initial conditions at 0 and at a nearby point, which in practice was 10^{-5}, deemed close enough that the Taylor approximation would be close enough). Since it is possible to scale the variables in order to satisfy the normalization and asymptotic flatness conditions (2.35) and (2.34), there is some freedom to choose arbitrary values for some of the parameters. Picking the mass parameter to be defined to be $m = \pm 1$ and assuming that $T(0) = 1$ (though of course T will only be 1 at infinity, in general, since it measures the "time dilation" factor at a point as measured by an observer at infinity), numerical solutions were found by fixing the parameter α_1 and getting numerical solutions in the independent variable ω, the energy of the field.

It was reported that these solutions were continuous in both α_1 and ω, which makes it reasonable to use this method (if this did not hold, the qualitative results being sought would not be expected to be obtained in this way, since properties of the solutions would not necessarily be extendible to nearby values of the parameters). The solutions had T going to a nonzero limit

τ at infinity (hence capable of being scaled), T and A everywhere positive, the spinor magnitude $\alpha^2 + \beta^2$ going to zero faster than order r^{-2}, $A \mapsto 1$ as $r \mapsto \infty$ (which is necessary for asymptotic flatness), and

$$\lim_{r \mapsto \infty} \frac{r}{2}(1 - A(r)) < \infty$$

which is the condition for finite ADM mass. Since every other essential condition can be met by scaling, we know this solution is at least admissible.

Further study of the solutions found revealed some important qualitative properties - properties which are revealed by study of the numerical solutions and can be assumed to hold for exact solutions since the numerically discovered ones will be sufficiently close (due to continuity of the solutions). First, it was noted that for positive mass (scaled to $m = 1$) any fixed value of α_1, there were a countable set of solutions for various values of ω from $\omega_0 < \omega_1 < \cdots < \omega_{max}$. The lowest ω corresponds to the *ground state* and the higher ω to *excited states* for the Dirac particle. The radial graphs of the functions A and T associated to these states were seen to have certain regularities: T is always a monotone decreasing function decaying from a value greater than 1 at $r = 0$ toward $T = 1$ as $r \mapsto \infty$ (as should be expected, since the mass should be expected to be concentrated at the center and thus cause time dilation relative to an observer at infinity, corresponding to a high T value, while at infinity, the metric is asymptotically flat - the monotonicity, however, is new). The A function, on the other hand, is not monotone: its exact behaviour depends on which excited or ground state the particle is in. It is equal to 1 at the origin, and asymptotically approaches 1 as $r \mapsto \infty$, but between these, it dips, and has some number of relative minima - one for the ground state, and for the nth excited state, $n + 1$ minima. For negative mass, similar properties were observed in both A and T.

We have briefly discussed the characteristics of the T and A curves in r - the remaining variables for which we get numerical results are α and β, the spinor components. Since these are the two components of a spinor, we can best understand the behaviour of the curves found as a parametric curve in $\alpha - \beta$ space. For every case, this curve is a closed curve beginning and ending at the origin (indicating asymptotic behaviour and initial state). For the ground state and small values for the initial value α_1, this curve stays in the first quadrant and has no self-intersections, while for higher values of α_1 it develops a "kink" and then, for still higher α_1, a self-intersection. For excited states, the curve no longer remains in the first quadrant: for the first excited state (and small α_1), it passes through all but the fourth (with no self-intersections) and for the second, it passes through all four, and does intersect itself in the first, resembling a cardioid. Somewhat similar phenomena are observed for the negative mass states. This illustrates that the excited states exhibit more complicated behaviour than the ground states, which is the same result noted in the case of the function A, for instance.

The next significant feature of the solutions (which, in part, leads to the investigation of stability features) appears when one examines the relationship, in any given ground or excited state, between mass m and energy ω, as parameterized by α_1. That is, considering the nth excited state, fixing the mass and varying α_1, one gets a one-parameter family of solutions having particular energy ω: rescaling m and ω to give physically meaningful results, one gets solutions only in a particular range of values of m, and the curve as a whole has a spiral shape: for a low value of m, there is a unique solution, but

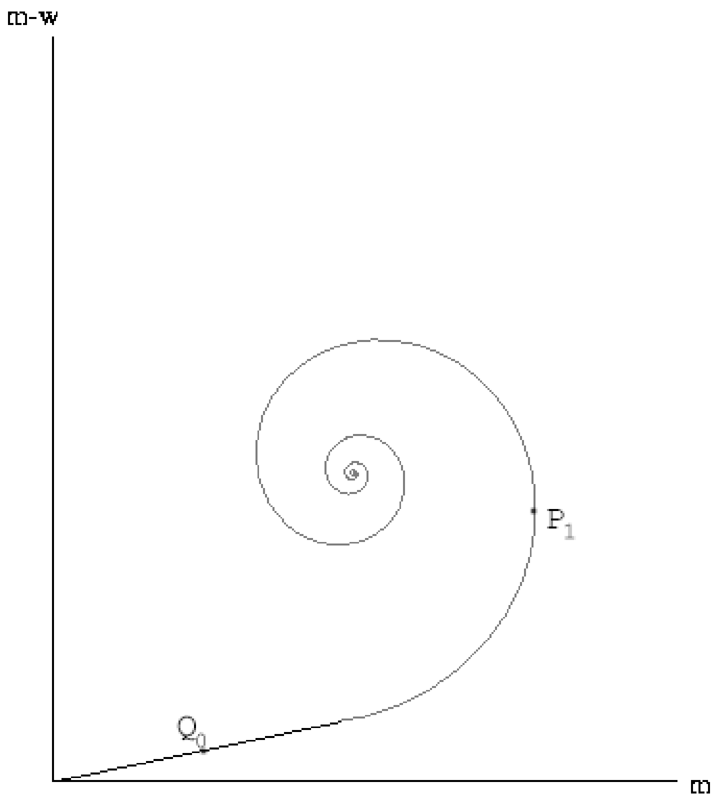

Figure 3.1: Qualitative Properties of Mass Spectrum Plot

as the curve (parameterized by α_1) continues, it reaches a maximum value of m for which solutions exist, then turns back and turns around a fixed point or cycle (as illustrated in figure 3.1.1) that for certain critical values of m there may be countably many energy states, while at others there will be a finite number. In every one of these solutions we have the energy less than the rest mass ($\omega < m$), which implies that we are looking at a system of fermions in a *bound* state: to separate the particles we would have to put in energy to bring the total energy up to the separate rest mass of a particle, since for two separated, noninteracting particles, the total energy will be the sum of their separate rest masses. These spirals, though quantitatively different for different excited states, appear (empirically) to be qualitatively the same for all ground and excited states for both positive and negative mass.

A similar spiral can be found by plotting $\rho - 2\omega$ (where ρ is the ADM mass). This quantity represents the energy contained in the gravitational field (since the ADM mass represents the total mass-energy of the system as measured from infinity, hence the total energy of the gravitational field and the Dirac particles themselves taken together - and we are looking at a pair of fermions bound together). It is negative for small m, meaning that the bound state has less energy than the unbound state, hence that energy must be put into the system to break apart the fermions, and thus suggesting that this state should be stable in this range. For the higher values of m the solution should then be unstable since it will release energy to decay into an unbound state, as indicated by the positive value of $\rho - 2\omega$. We have thus been led to the question of stability of the solutions.

3.2 Stability Analysis

3.2.1 Numerical Considerations

In order to judge the physical significance of the solutions we have found for the Einstein-Dirac equation, one of the questions it is natural to ask is whether the solutions represent a stable configuration. If small perturbations would disrupt a solution and render it unstable, we would not expect the corresponding physical configuration to occur naturally in physical situations. Therefore, we must consider the behaviour of our solutions under perturbation: some of this consideration is numerical, and some is topological in nature.

We consider spherically symmetric perturbations only, here - this has the simple effect of making the functions A and T in the form of the metric (2.1.1) dependent upon time t as well as radius r. The same methods can be used to calculate the Dirac operator, but a time derivative of A now enters into it. In contrast to (2.4), we thus have:

$$G = iG^t(\frac{\partial}{\partial_t} - \frac{\dot{A}}{4A}) + iG^r(\frac{\partial}{\partial r} + \frac{1}{r}(1 - \frac{1}{\sqrt{A}}) - \frac{\partial_r T}{2T}) + iG^\theta \partial_\theta + iG^\phi \partial_\phi$$

and we can separate the angular dependence by an ansatz of a form similar to that of (2.9), but (since the time dependence will be inside the spinor in the functions z which play the role of the radial u of (2.9)) of the form

$$\Psi = \frac{\sqrt{T}}{r} \begin{pmatrix} z_1(r,t)e_a \\ i\sigma^r z_2(r,t)e_a \end{pmatrix} \tag{3.1}$$

Following much the same procedure as before, this yields a time-dependent form of the Dirac equation as a 2-component ODE:

$$\begin{aligned} \Bigg(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(iT\partial_t - \frac{i}{4}\frac{T\dot{A}}{A} + \frac{i}{2}\dot{T} \right) \\ - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{A}\delta_r + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{1}{r} - m \Bigg) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \end{aligned} \tag{3.2}$$

On the other hand, the time-dependent form of the energy-momentum and Einstein tensors gives the dependent Einstein equations to be the following:

$$\frac{A-1}{r^2} + \frac{A'}{r} = -8\pi \frac{2iT^2}{r^2} Re(\overline{z_1}\partial_t z_1 + \overline{z_2}\partial_t z_2) \tag{3.3}$$

$$\begin{aligned} \frac{T^2\dot{A}}{rA} =& -8\pi Re\Big(\frac{iT^2}{r^2}(\overline{z_1}\partial_r z_1 + \overline{z_2}\partial_r z_2) \\ &+ \frac{T^3 A^{-\frac{1}{2}}}{r^2}(\overline{z_1}\partial_t z_2 + \overline{z_2}\partial_t z_1) \Big) \end{aligned} \tag{3.4}$$

$$\frac{A-1}{r^2} + \frac{2AT'}{rT} = 8\pi \frac{2TA^{\frac{1}{2}}}{r^2} Re(\overline{z_1}\partial_r z_2 + \overline{z_2}\partial_r z_1) \tag{3.5}$$

$$\begin{aligned} 8\pi \frac{2}{r^3} T Re(\overline{z_1} z_2) =& \frac{A'}{2r} - \frac{AT'}{rT} - \frac{A'T'}{2} + \frac{2AT'}{T^2} \\ & - \frac{AT''}{T} + \frac{3T^2\dot{A}^2}{4A^2} + \frac{T\dot{A}\dot{T}}{2A} + \frac{T^2\ddot{A}}{2A} \end{aligned} \tag{3.6}$$

Given values for the mass parameter m and energy parameter ω, and a compatible solution $(\alpha, \beta, A, T)(r)$ of the equations (2.22)-(2.25), we wish to consider the general form of a perturbation of this solution. An ansatz for this which specializes to the case already examined in the case of the time-independent case makes use of the redefined spinors α and β implicitly defined by expressing the z_i by

$$z_1(r,t) = e^{-i\omega t}\alpha(r,t)$$

and

$$z_2(r,t) = e^{-i\omega t}\alpha(r,t)$$

The perturbations in the spinors which we are considering are therefore small time-dependent deviations from the static (α, β, A, T) given. Since A and T are real and the spinors α and β are (generally) complex, we thus have the perturbation as:

$$\alpha(r,t) = \alpha(r) + \varepsilon(a_1(r,t) + ia_2(r,t)) \tag{3.7}$$

$$\beta(r,t) = \beta(r) + \varepsilon(b_1(r,t) + ib_2(r,t)) \tag{3.8}$$

$$A(r,t) = A(r) + \varepsilon A_1(r,t) \tag{3.9}$$

$$T(r,t) = T(r) + \varepsilon T_1(r,t) \tag{3.10}$$

(3.11)

where a_i, b_i, A_1 and T_1 are real-valued.

The analysis of this perturbation proceeds as follows: one substitutes the general perturbation into the Einstein-Dirac equations and assumes ε is small enough that all but the first order terms can be neglected. This gives a system of linear ODEs for the perturbing functions. An unstable solution would be one which admits the possibility of such a perturbation growing exponentially in time (since the equations are linear), so if we assume that time dependence is of this form and show that there are no nontrivial such solutions, then we will have shown the stability of the known solution. We thus assume that (for f representing in turn each of the functions a_j, b_j, A_1 and T_1) we have

$$f(r,t) = e^{\kappa t} f(r)$$

(noting that the same κ is used in every case since we are looking at linear perturbations).

The linear ODEs obtained are the following:

$$\begin{aligned}
\sqrt{A}a_1' &= \frac{a_1}{r} - (m+\omega T)b_1 + \kappa T b_2 - \frac{A_1}{2A}\big(\frac{\alpha}{r} - (m+\omega T)\beta\big) - \omega T_1 \beta \\
\sqrt{A}a_2' &= \frac{a_2}{r} - (m+\omega T)b_2 - \kappa T b_2 - \frac{A_1}{4A}T\beta - \kappa\frac{T_1}{2}\beta \\
\sqrt{A}b_1' &= -(m+\omega T)a_1 - \frac{b_1}{r} - \kappa T a_2 - \frac{A_1}{2A}\big(-(m+\omega T)\alpha - \frac{\beta}{r}\big) - \omega T_1 \alpha \\
\sqrt{A}b_2' &= -(m+\omega T)a_2 - \frac{b_2}{r} + \kappa T a_1 - \frac{A_1}{4A}T\alpha - \kappa\frac{T_1}{2}\alpha
\end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
2rAT_1' = \frac{A_1 T}{A} - T_1 + AT_1 + \frac{32\pi T^2}{r}(\alpha_1\beta + \beta_1\alpha) \\
+ 16\pi T^2\alpha(2ma_1 - 2\omega T a_1 + \kappa T a_2) \\
- 16\pi T^2\beta(2mb_1 + 2\omega T b_1 - \kappa T b_2) \\
- 16\pi T_1\big(3\omega T^2(\alpha^2+\beta^2) - \frac{4}{r}T\alpha\beta - 2mT(\alpha^2-\beta^2)\big) \\
+ 16\pi\frac{A_1 T}{A}\big(\omega T^2(\alpha^2+\beta^2) - \frac{2}{r}T\alpha\beta - mT(\alpha^2-\beta^2)\big)
\end{aligned}
\tag{3.13}
$$

In addition, one obtains a purely algebraic condition:

$$A_1 = 16\pi\frac{\sqrt{A}T}{\kappa r}(-(\kappa b_1 + 2\omega b_2)\alpha + (\kappa a_1 + 2\omega a_2)\beta)$$

In addition, there are initial conditions at $r = 0$ and the constraints demanding asymptotic flatness and the normalization condition to consider in this system. Initial conditions are given by a Taylor expansion, as before:

$$a_1(r) = a_{10} + \mathcal{O}(r^2) \tag{3.14}$$

$$a_2(r) = a_{20} + \mathcal{O}(r^2) \tag{3.15}$$

$$T_1(r) = T_{10} + \mathcal{O}(r^2) \tag{3.16}$$

$$b_j(r) = \mathcal{O}(r^2) \tag{3.17}$$

$$A_1(r) = \mathcal{O}(r^2) \tag{3.18}$$

while the normalization condition (2.26) (using the newly redefined α and β as the components of Φ) must still be satisfied. We note that the conservation of current means this integral is the same at all times, hence equal to its limit as $t \mapsto -\infty$, where the wavefunction approaches the unperturbed static solution.

It can be verified (using a symbolic computation program) that any solutions to the differential and algebraic equations in the above system satisfy the Einstein equation so we have a consistent system. To demonstrate stability of the solutions we have found (under first-order perturbation) it suffices to show nonexistence of such solutions for a value of the exponential growth rate parameter κ which are strictly positive (leading to an actual exponential growth of the perturbation).

In the following argument, we make several coordinate transformations to find a convenient way to examine the perturbational effects. If we first make a small reparametrization of time,

$$t \mapsto t - \varepsilon \frac{T_1(0)}{\kappa T(0)} e^{\kappa t}$$

we find that the form 3.7 of the perturbation remains the same, but the functions T_1, a_2 and b_2 change by a radial factor:

$$T_1(r) \mapsto T_1(r) - \frac{T_1(0)}{T(0)} T(r) \tag{3.19}$$

$$a_2(r) \mapsto a_2(r) - \frac{\omega}{\kappa} \frac{T_1(0)}{T(0)} \alpha(r) \tag{3.20}$$

$$b_2(r) \mapsto b_2(r) - \frac{\omega}{\kappa} \frac{T_1(0)}{T(0)} \beta(r) \tag{3.21}$$

(we note the common form of these transformations). By choosing a suitable reparameterization, we can thus fix, for example, T_1 at the origin to be zero so that $T_1(r) = \mathcal{O}(r)$ and thus reduce the number of free parameters which characterize the perturbed solution. This, however, weakens the asymptotic flatness condition 2.33 to a form more like 2.34, so that T_1 approaches, say μ in the limit as $r \mapsto \infty$. Further, we can eliminate a second parameter by noting that the linearity of the equations allows us to scale any solution, hence we may fix one more parameter by a multiplicative factor - say, set $a_{20} = 1$. This leaves only the parameter a_{10} to determine the solution.

To show stability, we recall, we must show that there are no solutions for this perturbed system having the form 3.12 for which κ is positive (since this would lead to runaway changes in the state, which indicates instability). Without entering into excessive detail in the numerics (which are not our current focus), we can briefly describe what this involves: in the a_1 vs. b_1 plots (as created numerically as solutions to our perturbed system), we are looking at perturbations from the $\alpha - \beta$ graph, so that the $a_1 - b_1$ graph is similar to it for small κ. This graph, plotting the spinor components against each other, gives a parametric curve with parameter r. Near $r = 0$ this is near the origin (both α and β are zero) and as $r \mapsto \infty$ it again returns to the origin in the unperturbed solutions, forming a closed curve (in the case of excited states, there may be self-intersections of this curve - for the ground state there are none). To show that the normalization integral cannot be finite, it suffices to show that the perturbed version of this graph is bounded away from zero for large time. This was initially difficult to judge (due to inaccuracies in the numerics) so, noting that the a_2 and b_2 are approximately multiples of α and β respectively, a transformation $\hat{a}_2 = a_2 - \mu\alpha$ and $\hat{b}_2 = b_2 - \mu\beta$ was used. Rewriting (3.12) in these new variables gave much improved accuracy. To solve the system, initial data at $r = 0$ were constructed (approximately) by finding initial values satisfing the property that $\lim_{r\mapsto\infty}(a_1, b_1)(r)$ is minimized, by choosing a cutoff value of r, R which minimize it (since beyond a certain value, inaccuracies in the numerical solutions will accumulate and make results unusable). By doing this for various values of κ, it was observed that, indeed, for positive κ these plots diverge very quickly away from zero for large r - though the good behaviour (similar form) of the $(\hat{a}_2, \hat{b}_2)(r)$ plots for these various values suggests that inaccuracies in the numerics are not responsible for the divergence, and thus that, indeed, for positive κ, the spinors are bounded away from zero (indeed diverge) and so the normalization integral cannot be finite, hence no such solution can exist. This would imply that the particle-like solutions found earlier would be stable. The same method applies to the excited states as to the ground state.

We point out here that these results only work for small mass m (weak coupling of the Dirac field to gravity), where the linearized equations are tractable. To deal with larger m, in the domain where the $|m|$ vs. ω, we must resort to topological methods involving the Conley index - which is described in Appendix B, and the use of which in this context is dealt with in the next section.

3.2.2 Topological Considerations

With the understanding of the Conley index developed in Appendix B, some illuminating results can be obtained regarding the stability of the class of solutions found previously for the Einstein-Dirac equation. We discussed in the previous section some of the numerical stability analysis which was effective for weak coupling of the Dirac field to the metric (that is, for small mass). The study of the mass-energy spectrum for higher m requires the topological results just described.

To do this, we regard the mass of the fermion to be the bifurcation parameter (that is, the main parameter for the Dirac equation - we consider the mass-energy spectrum curve (shown in figure 3.1.1) as representing fixed values of $m - \omega$ relative to m). The reason for this choice is not obvious, since the scaling factor means that m is not in fact fixed, while m and ω enter into the linear form of the field equations in exactly the same way. In fact, solutions to these linearized equations do not determine ω except up to a linear time-dependent perturbation (for more details on this, see [FSY1], appendix B), but m is entirely determined by the solution, since $G\Psi = m\Psi$ so that the inner product of $G\Psi$ with itself is just m^2.

Having adopted m as the relevant parameter, we then have a series of equations on $m - \omega$, corresponding to the mass-energy spectrum for the n^{th} excited state, and we can analyze these by continuation. For instance, near $m = 0$, we have a stable solution Q_0, which has Conley index Σ^0, the homotopy type of the pointed zero-sphere. The importance of continuation becomes clear here, for as we vary the parameter m, we can continue this stable solution all the way to the turning point P_1 of the spiral curve of the mass-energy spectrum, where the mass attains the critical value $m = m_1$. At this point, we have a degenerate solution (with flows entering any neighborhood on one side and exiting the other) so that the Conley index of the solution P_1 is just $\overline{0}$, namely the homotopy type of the pointed one-point space (since the neighborhood contracts down to the single exit point). Since, moving to lower m from this solution at $m = m_1$, we can construct neighborhoods containing both the fixed points which "bifurcate" from P_1, then the Conley index for a region containing both of these must be $\overline{0}$ - hence, since we know the bottom solution (being a continuation of Q_0) has flows only entering its neighborhood, the top solution must have Conley index Σ^1, the homotopy type of a pointed 1-sphere (circle). In other words, this solution is unstable.

This type of argument applies at every bifurcation point in the mass-energy spectrum's spiral - that is, every point P at which there is a degenerate solution, which corresponds to a vertical tangent to the curve. The Conley index of each of these points is $\overline{0}$, and so each of the pairs of solutions into which these degenerate solutions bifurcate must have this same combined Conley index, so there are, alternately, stable solutions with index Σ^0 and unstable solutions with index Σ^1.

Part II

Nonexistence of Black Hole Solutions

Chapter 4

Case I: Spherically Symmetric, Static EDM System

In this chapter, we consider for the first time a fully coupled system combining the Einstein, Dirac and Maxwell equations. This is a quite general configuration, since the Dirac equation describes the behaviour of a fermionic field, and Maxwell's equation describes a (in this case force-carrying) bosonic field: these are the two known classes of physically occurring fields. The fermionic field may be considered to represent a matter field composed of indistinguishable Dirac particles (for instance, electrons). These will be modeled as interacting through the electromagnetic field carried by the bosonic field, namely the photons represented by Maxwell's equation, as well as through a (non-quantized) gravitational field represented by the Einstein equations.

Recall that, when we showed the existence of particle-like solutions for the Einstein-Dirac equations in the first part of this work, we noted that, for any given state, for mass parameter above a critical value, solutions cease to exist. It has been shown in [FSY4] that this property also holds for the Einstein-Dirac-Maxwell system as well. The natural hypothesis is that this corresponds to the formation of black hole solutions: the center of mass of the system becomes a black hole, and thus the solution is no longer particle-like. In [FSY3] this was shown not to work under certain restricted conditions - such a black hole solution, given restricted symmetry requirements and minimal coupling, could not contain nonvanishing Dirac field. The results we shall be examining examine this question of black hole solutions further.

Our purpose in this chapter is to show that, in the restricted case of spherically symmetric, static solutions, there are no black hole solutions with nontrivial matter field outside the horizon - in other words, there are only the Reissner-Nordström solutions. This may be interpreted as stating that, if a cloud of Dirac particles (such as electrons) which is spherically symmetric collapses into a black hole preserving that symmetry, none of the matter can remain outside the horizon. This is an effect arising from the quantum-mechanical formulation of the fields in consideration, and does not occur in the classical case.

4.1 EDM Equations in Spherically Symmetric, Static System

4.1.1 Dirac Equation

The first step in examining the coupled EDM system will be to derive the form of the Dirac operator in the case of a spherically symmetric, static s-pacetime in which gravity is coupled to both the matter field governed by the Dirac equation and also to the electromagnetic field. This is a straightforward generalization of the form of the operator in the similar case without electromagnetism, which we examined in Part I. The only alteration in the form of the operator there derived is in the time coordinate. This arises in the following way: the electromagnetic field is described by the potential $\mathcal{A} = (-\phi, \vec{0})$ in the usual way. The Coulomb potential ϕ appears in this version of the operator, which is:

$$\begin{aligned} G &= iG^j\partial_j + B \\ &= iT\gamma^0(\partial_t - ie\phi) + \gamma^r\left(i\sqrt{A}\partial_r + \frac{i}{r}(\sqrt{A}-1) - \frac{i\sqrt{A}T'}{2T}\right) + i\gamma^\theta\partial_\theta + i\gamma^\phi\partial_\phi \end{aligned} \tag{4.1}$$

where the γ matrices are, as before, the Dirac matrices for flat spacetime in polar coordinates.

It is clear here (using the definition of the G^j) that the form of the Dirac operator is essentially the same as that obtained in (2.4), except that the term $iT\gamma^0(\partial_t)$ has become $iT\gamma^0(\partial_t - ie\phi)$. We shall therefore not elaborate upon the derivation, wherein the only difference would be an accounting for the electromagnetic potential.

Now we must consider some quantum mechanical features of Dirac fields in order to appreciate the behaviour of this system. To do this, we shall make a brief excursion to describe some quantum mechanics of particles with spin. First, since we will wish to consider solutions to the Dirac equation in terms of eigenstates of other operators, we shall develop these briefly. The first of these is the total angular momentum operator $\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2$, since we wish any solutions for the Dirac equation in this situation to be eigenvalues of this operator - in particular, we would wish $\mathbf{J}^2\Psi = 0$, corresponding to the eigenvalue 0, which should be the total angular momentum of the multiplet (multiparticle) state. This is an illustration of the physical meaning of the operators we are considering: a Hermitian operator on the state space of the system represents a physical observable (in this case the total angular momentum including spin components), and its eigenvalues represent the distinct quantum numbers which that observable value can attain. A physically observable STATE corresponds to an eigenstate (eigenfunction or eigenvector, depending on how we think of the elements of the Hilbert space of states of the system) of the operator: states which are not eigenstates do not correspond to classical states of the system, but rather to linear combinations, or superpositions of them. We now develop briefly the operator **J**, following roughly the treatment in [Sch].

The total angular momentum operator combines two components: the first, **L**, corresponds to classical angular momentum, which we may think of as representing the rotation of the Dirac particles about a center of motion, which in our case will be the point $r = 0$; the second component, **S** is the *spin angular momentum* and is a feature which does not arise in classical mechanics. It is a result of the fact that the Dirac field is a spinor field and has some internal freedom. Since the spin group is the universal covering group of the rotation group in three dimensions, and thus these two Lie groups have the same Lie algebra, it follows that infinitesimal elements of each can be added. So the total angular momentum will take account of both. The operator **L**, the angular momentum about the origin for a particle, is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the (vector) operator for the observable representing the position of the particle relative to the origin, and $\mathbf{p}$ is the momentum operator. It is an infinitesimal rotation about the origin - that is, an element of the Lie algebra of the rotation group $SO(3)$), which Lie algebra has three generators, representing the infinitesimal rotations about the x, y and z axes (in Cartesian coordinates). We designate these three generating operators by L_x, L_y and L_z, respectively. We remark that the fact that we are considering a rotationally symmetric system corresponds to the claim that these operators commute with the Hamiltonian H describing the evolution of the system, since these are then constants of motion.

The *spin angular momentum* operator $\mathbf{S}$ measures the change in spin components - since this is an element in the same Lie algebra as the angular momentum, it is reasonable to consider it an angular momentum as well, although it is an infinitesimal element of a different Lie group. We consider a rotation not only to rotate the particle in space but also to rotate its spin components, so that the total rotation is represented by the operator $\mathbf{J} = \mathbf{L} + \mathbf{S}$. The separate components of the total angular momentum are not conserved quantities of the evolution, since in general they will not commute with the Hamiltonian, whereas $\mathbf{J}$ will. In other words, in classical terms, it is the total angular momentum which is conserved - so that we might think of angular momentum being transferred from the (classical) rotation to the spin momentum of a particle, with the total sum being conserved. This is then the appropriate operator to consider as angular momentum. More particularly, we will be considering the operator $\mathbf{J}^2$: this can be found (cf. [Sch]) to have eigenvalues of the form $j(j+1)$, where j is the angular momentum quantum number. In the situation we are considering, this is the angular momentum of each Dirac particle, and takes on half-integral values, so that

$$j = \frac{1}{2}, \frac{3}{2}, \ldots$$

We are also interested, since we are coupling our Dirac particles to the electromagnetic field, in the component of angular momentum about the axis defined by the field lines, which we are designating the z axis. This is the J_z operator, which possesses the convenient property that it commutes with $\mathbf{J}^2$. Since it represents the component of angular momentum along a particular axis, it can take eigenvalues with absolute value at most j, but can otherwise take on any half-integral value (representing, therefore, an angular momentum just as $\mathbf{J}$ does), so that we have its eigenvalues to be k where:

$$k = -j, -j+1, \ldots, j-1, j$$

Both the $\mathbf{J}_z$ and $\mathbf{J}^2$ operators commute with the Dirac operator, as does the time-translation operator $i\partial_t$ and the operator $\gamma^0 P$ where P is the parity operator.

Since the four operators $\mathbf{J}$, J_z, $i\partial_t$ and $\gamma^0 P$ all commute with the Dirac operator and all commute with each other, eigenstates of the Dirac equation will also be eigenstates of each of these operators, since commuting operators can be simultaneously diagonalized, and we have chosen a four-dimensional representation for the spinor state, so that any solution for the Dirac equation can be written as a linear combination of simultaneous eigenstates for these four operators. That is, if $(D-m)\Psi = 0$ we have:

$$\begin{aligned} i\partial_t \Psi &= \omega \Psi \\ \mathbf{J}^2 \Psi &= j(j+1)\Psi \\ J_z \Psi &= k\Psi \\ \gamma^0 P \Psi &= \pm\Psi \times \begin{cases} 1 & \text{for } j+\frac{1}{2} \text{ even} \\ -1 & \text{for } j+\frac{1}{2} \text{ odd} \end{cases} \end{aligned}$$

We may thus index solutions according to the eigenvalues to which they correspond for each of these operators. The spectrum of the two angular momentum operators has already been described, consisting of discrete half-integral eigenvalues. The time-translation operator has a continuous spectrum with all real values possible as eigenvalues, so $\omega \in \mathbb{R}$ is all we can say. The eigenvalues for the parity operator are ± 1, so we can index the simultaneous eigenstates of these operators by the eigenvalues corresponding to them. These form a basis for the solution space which we indicate

$$\Psi^c_{jk\omega} \text{ where } c = \pm \ , \ j = \frac{1}{2}, \frac{3}{2}, \dots \ , \ k = -j, -j+1, \dots, j \ , \ \omega \in \mathbb{R}$$

so that the previous set of equations becomes

$$\begin{aligned} i\partial_t \Psi^c_{jk\omega} &= \omega \Psi^c_{jk\omega} \\ \mathbf{J}^2 \Psi^c_{jk\omega} &= j(j+1)\Psi^c_{jk\omega} \\ J_z \Psi^\pm_{jk\omega} &= k \Psi^\pm_{jk\omega} \\ \gamma^0 P \Psi^\pm_{jk\omega} &= \pm \Psi^\pm_{jk\omega} \times \begin{cases} 1 & \text{for } j + \frac{1}{2} \text{ even} \\ -1 & \text{for } j + \frac{1}{2} \text{ odd} \end{cases} \end{aligned} \tag{4.2}$$

Now for each solution of the Dirac equation which is one of these basis states, the Dirac operator can be reduced to a system of ODEs in the variable r much as was done in the previous part in which we reduced the Dirac equation through symmetries. Note that this refers to a solution for a single particle, however: it now becomes necessary to construct multiparticle solutions from those we have found for single particle states.

Whereas the state for a single particle - the wavefunction - is represented as a point in a Hilbert space $\mathcal{H}$, it must follow that multiparticle states representing n particles are represented by points in the tensor product of n copies of $\mathcal{H}$, namely $\mathcal{H}^n = \mathcal{H} \otimes \mathcal{H} \cdots \otimes \mathcal{H}$, which is again a Hilbert space. An important point is the question of how this multiplet state vector behaves under interchange of the particles composing it. That is, if we have the multiplet state as a function $\Psi(\Psi_1, \Psi_2, \ldots, \Psi_n)$ of n single-particle states, what is the change in the multiplet state if we permute the entries - what is $\Psi(\Psi_{\sigma(1)}, \Psi_{\sigma(2)}, \ldots, \Psi_{\sigma(n)})$, for some permutation $\sigma \in \mathfrak{S}_n$? As Weyl remarks in [Weyl] (p240), we would expect any physically realistic situation to be confined to either the totally symmetric or totally antisymmetric subspace of the Hilbert space $\mathcal{H}^n$. This is first because with a hermitian evolution, any configuration which begins in either of these two spaces (which decompose $\mathcal{H}^n$ completely) will remain there, and second because we expect that, for identical particles, interchanging the states of two particles should represent essentially the same physical situation, hence be represented by a scalar multiple of Ψ of the same magnitude. The only question is whether it should be symmetric or antisymmetric in σ. In the case of particles described by the Dirac equation, *fermions*, it will be antisymmetric, so that we have an element of the antisymmetric tensor algebra of $\mathcal{H}$ (note that since $\mathcal{H}$ is infinite-dimensional, there is no limit in principle to the number of particles). We thus represent the multiplet state by the so-called *Hartree-Fock state*:

$$\Psi^{HF} = \Psi_1 \wedge \Psi_2 \wedge \cdots \wedge \Psi_n$$

We remark briefly here that it is this representation for the combined multiparticle state which gives rise to the *Pauli exclusion principle*, which asserts that two fermions (Dirac particles, for our purposes) may not have the same quantum state. This principle was predicted empirically on the basis of observations of the filling of electron shells in the periodic table of elements (noting that there are at most two electrons in any energy state, one in each parity), but was explained by the antisymmetry of the multiparticle wave function for fermions: the Hartree-Fock state is zero if any of its components are scalar multiples of each other - that is, represent the same state.

In the case we are considering, we are interested in combining the possible values k of the z-axis component of angular momentum (the component along the axis of the electromagnetic field). So we are actually considering

$$\Psi^{HF} = \Psi^c_{j(k=j)\omega} \wedge \Psi^c_{j(k=j-1)\omega} \wedge \cdots \wedge \Psi^c_{j(k=-j)\omega} \tag{4.3}$$

We observe that, by the fact that these Ψ are eigenstates of J_z with eigenvalue k (as in equation 4.2), we in fact have the combined state as an eigenvalue of the operator extended in the usual way to $\mathcal{H}$, namely:

$$\begin{aligned} J_z\Psi^{HF} = {} & \left(J_z\Psi^c_{j(k=j)\omega}\right) \wedge \Psi^c_{j(k=j-1)\omega} \wedge \cdots \wedge \Psi^c_{j(k=-j)\omega} \\ & + \Psi^c_{j(k=j)\omega} \wedge \left(J_z\Psi^c_{j(k=j-1)\omega}\right) \wedge \cdots \wedge \Psi^c_{j(k=-j)\omega} \\ & + \cdots + \Psi^c_{j(k=j)\omega} \wedge \Psi^c_{j(k=j-1)\omega} \wedge \cdots \wedge \left(J_z\Psi^c_{j(k=-j)\omega}\right) \end{aligned} \tag{4.4}$$

that is, J_z acts on each component of the Hartree-Fock state - but these, being eigenstates, simply contribute a scalar multiple of the whole state:

$$\begin{aligned} J_z\Psi^{HF} = {} & \left(j\Psi^c_{j(k=j)\omega}\right) \wedge \Psi^c_{j(k=j-1)\omega} \wedge \cdots \wedge \Psi^c_{j(k=-j)\omega} \\ & + \Psi^c_{j(k=j)\omega} \wedge \left((j-1)\Psi^c_{j(k=j-1)\omega}\right) \wedge \cdots \wedge \Psi^c_{j(k=-j)\omega} \\ & + \cdots + \Psi^c_{j(k=j)\omega} \wedge \Psi^c_{j(k=j-1)\omega} \wedge \cdots \wedge \left((-j)\Psi^c_{j(k=-j)\omega}\right) \end{aligned} \tag{4.5}$$

But this reduces to just:

$$J_z\Psi^{HF} = \sum_{k=-j}^{j} k\Psi^{HF} = 0 \tag{4.6}$$

so that, in fact, that the Hartree-Fock state Ψ^{HF} is also an eigenstate of the angular momentum operator J_z, and in fact has no angular momentum about the z axis.

It is somewhat less easy to show that the Hartree-Fock state has total angular momentum also equal to zero (i.e. that it is spherically symmetric), and to do this we must use the so-called "Ladder Operators" $J_{\pm}$. These are rather similar to the a and a^t operators developed in the usual treatment of the harmonic oscillator (e.g. in [Coh]). A treatment of the Ladder operators for angular momentum can be found in [Coh] p. Although they do not represent classical observables, they represent (chapter VI, part C), and a situation similar to the current case is treated in [FSY3] using only the angular momentum $\mathbf{L}$. The addition or removal of a quantum of energy, or photon (in the case of the harmonic oscillator) or a quantum of angular momentum (in the present case). These operators are defined by:

$$J_{\pm} = J_x \pm iJ_y$$

so that $[J_+, J_-] = 2\hbar J_z$ (since $[J_x, J_y] = i\hbar J_z$). We use units where $\hbar = 1$ throughout, and thus $[J_+, J_-] = 2J_z$. This is analogous to the harmonic oscillator, where the corresponding operators are $\frac{1}{\sqrt{2}}(\mathbf{p} \pm i\mathbf{q})$ (where $\mathbf{p}$ is the position and $\mathbf{q}$ the momentum operator), whose commutator is 1. The ladder operators are clearly adjoints of each other, and we have the relations

$$\mathbf{J}^2 = J_+J_- + J_z^2 + J_z = J_-J_+ + J_z^2 - J_z \tag{4.7}$$

(which follows from the commutation relation above). From these, we can find that, as in the case of the harmonic oscillator, the $J_{\pm}$ act as "ladder" operators in the sense that they take an eigenvector associated with one eigenvalue to one associated with the next (or previous) eigenvalue. In the case of the harmonic oscillator (see e.g. [Coh]) this is given the interpretation of a creation or destruction operator. Here it takes a state into a state with one more quantum unit of angular momentum, with a multiplicative factor: we have:

$$J_{\pm}\Psi^{c}_{jk\omega} = \sqrt{j(j+1) - k(k \pm 1)}\Psi^{c}_{j(k\pm 1)\omega} \tag{4.8}$$

(This factor is clearer if we note that $j(j-1) - k(k+1) = (j-m)(j+m+1)$ and $j(j-1) - k(k-1) = (j-m+1)(j+m)$). But then of course if we apply this to the Hartree-Fock state in the usual way, we will obtain a series of terms which contain two copies of the same state, and since the wedge product is antisymmetric, each of these terms is zero, so that:

$$\begin{aligned} J_{\pm}\Psi^{HF} &= (K_j \Psi^{c}_{j(k=j)\omega}) \wedge \Psi^{c}_{j(k=j-1)\omega} \wedge \cdots \wedge \Psi^{c}_{j(k=-j)\omega} \\ +\Psi^{c}_{j(k=j)\omega} &\wedge (K_{j-1}\Psi^{c}_{j(k=j-1)\omega}) \wedge \cdots \wedge \Psi^{c}_{j(k=-j)\omega} + \cdots = 0 \end{aligned} \tag{4.9}$$

where the K_k are the appropriate scalars. In each case, there is a doubled term, corresponding to a physical system in which two Dirac particles are in the same state, which the Pauli exclusion principle (the antisymmetry of the $\wedge$-product) rules out. By 4.9 and 4.6, combined with the relation 4.7 decomposing the total angular momentum in terms of the ladder operators and J_z, we have that the Hartree-Fock state Ψ^{HF} is an eigenstate of the total angular momentum operator with eigenvalue 0, and thus that the Hartree-Fock state can be spherically symmetric, since

$$\mathbf{J}\Psi^{HF} = 0$$

and $\mathbf{J}$ is the infinitesimal generator of revolutions.

Thus, the multiplet state of $2j+1$ particles can be spherically symmetric even though each one may have nonzero angular momentum.

Having established that we may have a system of $2j+1$ Dirac particles which is static and spherically symmetric, we wish to separate out the time and angular dependence in the Dirac equation to simplify the computations. To do this, we choose an explicit ansatz for the wave functions, which will involve the spherical harmonics. These are simultaneous eigenfuncions of the operators $\mathbf{L}^2$ and L_z (which is why we make use of them in our ansatz), and are well known, but we shall discuss them here briefly, though more detailed treatment is found in ([Coh] chapter VI, part D).

The spherical harmonics are functions $Y_j^k(\theta, \phi)$ corresponding to the eigenvalues $j(j+1)$ for the $\mathbf{L}^2$ operator and k for the L_z operator. (Since we have not included spin angular momentum here, it is necessary to adjust this when dealing with the $\mathbf{J}^2$ and J_z operators, since now we have half-integral values, so we will have, for instance,

$$Y_{j+\frac{1}{2}}^{k+\frac{1}{2}}(\theta, \phi)$$

as a possible spherical harmonic). Direct calculations show that the θ and ϕ dependence in the Y can be separated out to give, for any particular index values α and β

$$Y_\beta^\alpha(\theta, \phi) = F_\beta^\alpha(\theta) e^{i\alpha\phi}$$

In particular, we can use the ladder operators J_- to construct spherical harmonics from the "maximum" where $j = k$, since the ladder operator takes an eigenstate corresponding to one eigenvalue to that corresponding to the next lowest quantum state (i.e. the next lowest of the discrete eigenvalues of the spectrum of the angular momentum operator). The maximum case is obtainable by writing the operators for which we want to find simultaneous eigenstates as ODE's, and is just:

$$Y_\alpha^\alpha = c_\alpha (\sin\theta)^\alpha e^{i\alpha\phi}$$

(this is unique up to a constant which is fixed for all the functions Y_α^α, $Y_\alpha^{\alpha-1}$, $\ldots$ $Y_{-\alpha}^\alpha$ obtained from it).

We define the 2-spinors $\chi_{j\pm\frac{1}{2}}^k$ by:

$$
\begin{aligned}
\chi_{j-\frac{1}{2}}^k &= \sqrt{\frac{j+k}{2j}} Y_{j-\frac{1}{2}}^{k-\frac{1}{2}} \begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{j-k}{2j}} Y_{j-\frac{1}{2}}^{k+\frac{1}{2}} \begin{pmatrix}0\\1\end{pmatrix} \\
\chi_{j+\frac{1}{2}}^k &= \sqrt{\frac{j+1-k}{2j+2}} Y_{j+\frac{1}{2}}^{k-\frac{1}{2}} \begin{pmatrix}1\\0\end{pmatrix} - \sqrt{\frac{j+1+k}{2j+2}} Y_{j-\frac{1}{2}}^{k+\frac{1}{2}} \begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}
\tag{4.10}
$$

(where j takes on half-odd-integral values and k runs in each case from $-j$ to j). The χ are eigenvalues of $K = \vec{\sigma}\vec{J} + 1$:

$$
\begin{aligned}
K\chi_{j\pm\frac{1}{2}}^k &= \begin{pmatrix} L_z + 1 & L_- \\ L_+ & -L_z + 1 \end{pmatrix} \chi_{j\pm\frac{1}{2}}^k \\
&= \mp(j + \frac{1}{2})\chi_{j\pm\frac{1}{2}}^k
\end{aligned}
\tag{4.11}
$$

In addition, multiplying by the "polar Pauli matrix" σ^r gives, by a simple direct calculation:

$$
\begin{aligned}
K\sigma^r\chi^k + j \pm \frac{1}{2} &= -\sigma^r K\chi_{j\pm\frac{1}{2}}^k \\
&= \pm(j + \frac{1}{2})\chi_{j\pm\frac{1}{2}}^k
\end{aligned}
$$

so that multiplication by the spinor σ^r interchanges the two integral-index forms corresponding to a half-integral j - in other words:

$$
\sigma^r\chi_{j\pm\frac{1}{2}}^k = \chi_{j\mp\frac{1}{2}}^k \tag{4.12}
$$

Note that for the values of χ we are simply associating, for each value of k, the spherical harmonics (multiplied by the appropriate spinor value) for the integral values on either side of k, added, leaving only one half-integral index, j. The whole ansatz, to get the 4-spinor values associated to the half-integral indices j and k will combine both the integral indices corresponding to both j and k. In particular, analogously to the development of the ansatz 2.9 for the solutions to the Dirac-Maxwell equation developed in section 2.1.2, we see that we have two 2-spinors interchangeable through multiplication by σ^r, motivating the form for the ansatz:

$$\Psi^+_{jk\omega} = e^{i\omega t}\frac{\sqrt{A}}{r}\begin{pmatrix} \chi^k_{j-\frac{1}{2}}\Phi^+_{jk\omega 1} \\ i\chi^k_{j+\frac{1}{2}}\Phi^+_{jk\omega 2}\end{pmatrix} \tag{4.13}$$

and

$$\Psi^-_{jk\omega} = e^{i\omega t}\frac{\sqrt{A}}{r}\begin{pmatrix} \chi^k_{j+\frac{1}{2}}\Phi^-_{jk\omega 1} \\ i\chi^k_{j-\frac{1}{2}}\Phi^-_{jk\omega 2}\end{pmatrix} \tag{4.14}$$

(note the difference between the two forms, indexed by parity: the interchange of the roles of the higher and lower integral indices on the χ). Here, the $\Phi^c_{jk\omega_i}$ are unknown radial functions $\Phi^c_{jk\omega_i}(r)$. This is an ansatz for a simultaneous eigenstate of the four operators $i\partial_t$, $\mathbf{J}^2$, J_z and $\gamma^0 P$ as in 4.2, and for such states, with this ansatz and the symmetry properties, we can (much as in the existence proof of part I) reduce the Dirac equation $(G-m)\Psi^c_{jk\omega} = 0$ to an system of ordinary differential equations, which in this case turn out to be these:

$$\sqrt{A}\frac{d}{dr}\Phi^\pm_{jk\omega} = \left[(\omega - e\phi)T\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} \pm \frac{2j+1}{2r}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} - m\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\right]\Phi^\pm_{jk\omega} \tag{4.15}$$

We remark here briefly on the similarity of this equation to that found in 2.4 - in the case where $j = \frac{1}{2}$ and the electromagnetic potential ϕ vanishes, we retrieve 2.4 from 4.15 immediately, since this is the case of a single Dirac particle coupled only to gravity.

It is this last (4.15) which we shall use for purposes of calculations, bearing in mind that solutions to the Dirac equation are linear combinations of such eigenstates.

As we have already discussed in part I, only normalized wavefunctions have physical meaning (since their squared norm represents a probability density, and must have total integral equal to 1). In physical situations, we therefore normalize the wavefunctions by their integral which, as we have already remarked, can be found by integrating on any spacelike hypersurface, since this is invariant when the wavefunction is a solution of the Dirac equations. So our normalization integral 2.6 becomes

$$\int_{\mathcal{H}} \overline{\Psi^c_{jk\omega}} G^j \Psi^c_{jk\omega} \nu_j d\mu_{\mathcal{H}} \tag{4.16}$$

where the terms are defined as before. In this case, however, the condition which we must place on this normalization integral is different: where previously we required that it be finite, the presence of a black hole in this case (and the fact that we are considering only the spacetime outside the horizon) makes it possible that contributions from inside the horizon might cancel contributions from the outside. This is because this integral, inside the horizon, becomes negative (since inside the horizon spacelike and timelike paths exchange roles). The fact that, near the horizon, there may be diverging positive and negative contributions canceling each other means that we can only expect to make the normalization integral be finite away from the horizon. That is, for every $r_0 > \rho$, we will have the above integral be finite when it is taken on the part of a hypersurface strictly outside $r = r_0$.

4.1.2 Einstein and Maxwell Equations

Having obtained the form in the previous section for the Dirac equation and the wavefunctions of the Dirac particles, we need to find the total current (for coupling to Maxwell's equations as the source of the electromagnetic field) and the energy momentum tensor of the matter field (for coupling to Einstein's equations as the source of the gravitational field). Using the explicit form for the wave equations (4.15), we find that the total current for the Dirac field is just the sum of the separate currents for the singlet states. More generally we would have

$$j^a = \overline{\Psi^{HF}} G^a \Psi^{HF}$$

since we are obtaining the current of the Dirac field through the observable represented by the operator G. This form, given the explicit ansatz we have chosen, reduces to

$$j^a = \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} G^a \Psi^c_{jk\omega}$$

because the antisymmetrization in the Hartree-Fock state has no effect since the different Ψ obtained for different values of k are orthogonal with respect to the bilinear form G^a on $\mathcal{H}$, as can be verified by a direct calculation using the form for the wavefunctions and the operator in terms of the $\Phi^c_{jk\omega}$.

Now because we are dealing with spherically symmetric solutions, it is clear that the components j^θ and j^ϕ of the current will be zero, since any nonzero current in these directions would break the spherical symmetry.

To find the time-coordinate component j^t, we make use of an identity on the 2-spinors $\chi^k_{j\pm\frac{1}{2}}$, namely that

$$\sum_{k=-j}^{j} \overline{\chi^k_{j\pm\frac{1}{2}}(\theta,\phi)}\chi^k_{j\pm\frac{1}{2}}(\theta,\phi) = \frac{2j+1}{4\pi} \tag{4.17}$$

(which follows from the constants involved in the definition of the χ and the norms of the spherical harmonics - summing over k we obtain this simple form). Using this identity, we can find that the component j^t has the form:

$$\begin{aligned} j^t &= \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} G^t(x) \Psi^c_{jk\omega} \\ &= \frac{T^2}{r^2}(\alpha^2+\beta^2)\frac{2j+1}{4\pi} \end{aligned} \tag{4.18}$$

where we recall that α and β are the real components of the unknown functions $\Phi^c_{j\omega}$. We remark that finding such real components can be done for the following reason.

The radial flux of the Dirac fields,

$$F(r) \cong \overline{\Phi^c_{j\omega}(r)} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Phi^c_{j\omega}(r)$$

is constant, since the flux integral over the boundary of any annular region about $r=0$ is zero (by conservation of current), and by symmetry. The Dirac equation thus implies that the radial flux is constant, while $|\Phi^c_{j\omega}|^2 \geq F$, so since the metric is asymptotically flat (and hence $|\Phi^c_{j\omega}|^2 \mapsto 0$ as $r \mapsto \infty$) we must have the radial flux to be zero. But this means that $\overline{\Phi^c_{j\omega 1}}\Phi^c_{j\omega 2}$ must have vanishing imaginary part. Since this is real, both components have the same phase, which we can arrange to be zero by a constant phase transformation (which is permissible by gauge freedom). Then, since the Dirac equation has real coefficients, all the spinors can be made real, justifying our use of α and β as real values for the spinor components.

In order to derive the energy-momentum tensor, we use the same variational technique applied in part I. Taking the variation of the Dirac action here, we find that in this case the energy momentum tensor is simply given by the norm of the wavefunction relative to the symmetrized bilinear form $iG_a(\partial_b - ie\mathcal{A}_b)$, namely:

$$T_{ab} = \frac{1}{2} Re \sum_{k=-j}^{j} \overline{\Psi^c_{jk\omega}} (iG_a(\partial_b - ie\mathcal{A}_b) + iG_b(\partial_a - ie\mathcal{A}_a))\Psi^c_{jk\omega} \tag{4.19}$$

From this it is relatively easy to compute the components of $\mathbf{T}$ involving t and r as indices. First, we observe that the mixed terms involving the angular coordinates vanish, so we do not need to compute these directly. This is because of spherical symmetry, which means that there is no stress along the angular directions. Directly performing the calculation of the stress-energy tensor using the ansatz 4.13, 4.14 and the Dirac operator 4.15 allows this to be found quite easily, making use of the fact (4.17) that the sum of the norms of the χ's over all k values is known to be $\frac{2j+1}{4\pi}$. This simplifies the expressions for the first two diagonal entries of the stress-energy tensor to:

$$T^t_t = \frac{(\omega - e\phi)T^2}{r^2}(\alpha^2 + \beta^2)\frac{2j+1}{4\pi} \tag{4.20}$$

and the second will be, depending on the parity of the state under consideration, where $c = \pm$,

$$T^r_r = -\frac{(\omega - e\phi)T^2}{r^2}(\alpha^2 + \beta^2)\frac{2j+1}{4\pi} \pm \frac{T}{r^3}\alpha\beta\frac{(2j+1)^2}{4\pi} + \frac{mT}{r^2}(\alpha^2 - \beta^2)\frac{2j+1}{4\pi} \tag{4.21}$$

Now to find T^θ_θ and T^ϕ_ϕ, we first note that due to spherical symmetry, they must be equal (since both simply represent an angular component of the stress-energy tensor, which, being spherically symmetric, has all angular components equal). They will be found by contracting the wave function with the angular part of the Dirac operator to find their sum, so that:

$$T_\theta^\theta + T_\phi^\phi = Re \sum_{k=-j}^{j} \overline{\Psi_{jk\omega}^c}(iG^\theta \partial_\theta + iG^\phi \partial_\phi)\Phi_{jk\omega}^c$$

but we have that these angular derivatives can be written in terms of the angular momentum as:

$$G^\theta \partial_\phi + g^\phi \partial_\phi = -\frac{1}{r}\sigma^r \vec{\sigma}\vec{L}$$

($\vec{\sigma}\vec{L}$ is as represented in the expression for the operator K above). With this equation, we can use the explicit ansatz for the wave functions and derive the angular components of $\mathbf{T}$ in terms of the spinors χ. In particular, we find them to be:

$$T_\theta^\theta + T_\phi^\phi = \alpha\beta\frac{T}{r^3} Re \sum_{k=-j}^{j} \left(\overline{\chi_{j\pm\frac{1}{2}}^k}\vec{\sigma}\vec{L}\chi_{j\pm\frac{1}{2}}^k - \chi_{j\mp\frac{1}{2}}^k\vec{\sigma}\vec{L}\chi_{j\mp\frac{1}{2}}^k\right)$$

(where we have used $\chi_{j\pm\frac{1}{2}}^k\sigma^r = \chi_{j\mp\frac{1}{2}}^k$). But we remark here that the operator $\vec{\sigma}\vec{L}$ has the χ as eigenvectors, because of their construction from the spherical harmonics (eigenvectors of $\vec{L}$) in linear combinations which preserve this property after application of $\vec{\sigma}$.

This finally yields:

$$T_\theta^\theta = T_\phi^\phi = \mp\alpha\beta\frac{T}{r^3}\left(\frac{(2j+1)^2}{8\pi}\right)$$

Once we have these components for the stress energy tensor, we can substitute these into the Einstein equations

$$G_j^i = R_j^i - \frac{1}{2}R\delta_j^i = -8\pi T_j^i$$

These field equations give us equations for A and T:

$$rA' = 1 - A - 2(2j+1)(\omega - e\phi)T^2(\alpha^2 + \beta^2) = r^2AT^2|\phi'|^2 \tag{4.22}$$

and

$$\begin{aligned} 2rA\frac{T'}{T} &= A - 1 - 2(2j+1)(\omega - e\phi)T^2(\alpha^2+\beta^2) \\ &\qquad \pm 2\frac{(2j+1)^2}{r}T\alpha\beta + 2(2j+1)mT(\alpha^2 - \beta^2) + r^2AT^2|\phi'|^2 \end{aligned} \tag{4.23}$$

With the Dirac current as source for Maxwell's equations

$$\nabla_l F^{kl} = 4\pi e j^k$$

we obtain a second-order equation for ϕ:

$$r^2A\phi'' = -(2j+1)e(\alpha^2+\beta^2) - \big(2rA + r^2A\frac{T'}{T} + \frac{r^2}{2}A'\big)\phi' \tag{4.24}$$

Combining these with the Dirac equations for α and β:

$$\sqrt{A}\alpha' = \pm\frac{2j+1}{2r}\alpha - ((\omega - e\phi)T + m)\beta \tag{4.25}$$

and

$$\sqrt{A}\beta' = ((\omega - e\phi)T + m)\alpha \mp \frac{2j+1}{2r}\beta \tag{4.26}$$

we obtain the complete set of Einstein-Dirac-Maxwell equations for the case we are considering. Here, we have $c = \pm$ as usual.

The normalization condition obtained in (4.16) then becomes, substituting these new forms,

$$\int_{r_0}^{\infty}(\alpha^2+\beta^2)\frac{\sqrt{T}}{A}dr < \infty \tag{4.27}$$

(for any $r_0 > \rho$).

We note the similarity of these equations (4.22)-(4.26) to those in part I, namely (2.22)-(2.25), which they become in the case where we remove the Maxwell equation and set $j = \frac{1}{2}$ and $\phi = 0$. A similar analysis of these equations would show that this system has particle-like solutions having properties similar to that system, but here we are interested primarily in the nonexistence of certain classes of solutions, and more specifically certain classes of black hole solutions. We discuss these results next.

4.2 Nonexistence Theorems

4.2.1 Characterization of Black Hole Solutions

With the computational framework we have now established for the fully coupled Einstein-Dirac-Maxwell equations in the spherically symmetric, static case, we can proceed to prove the theorems in which we are currently interested for this case. These will be nonexistence results for certain types of black hole solutions (namely solutions for which the Dirac field is non-vanishing). The standard definition of a black hole (e.g. [Hawk] p315) is that of a region of spacetime from which light or particles cannot escape - in other words, which is closed under the operation of taking the union with any future-directed timelike or null paths from any point in the region. For our purposes, we are interested in describing the fields on spacetime outside of such a black hole - and in particular, since we are dealing with a spherically symmetric spacetime, this amounts to defining our spacetime as the product of $\mathbb{R}$ (time) with a region outside a ball of some radius about the origin (a typical spacelike hypersurface). Study of normal such black hole solutions, the Schwarzchild and Reissner-Nordström solutions, lead us to characterize our understanding of "black hole solution" in the following way, which we will take as definitive in the present context. We will then prove that the two mentioned solutions are the only ones satisfying the given conditions.

We assume that spacetime is asymptotically flat far from the black hole, so that as $r \mapsto \infty$ we have $A(r) \mapsto 1$ and $T(r) \mapsto 1$. We characterize the *event horizon* of the black hole, the surface at $r = \rho$, by saying that as $r \mapsto \rho$ from above, we have $A(r) \mapsto 0$ and $T(r) \mapsto \infty$. This corresponds to the observational properties that near the event horizon of black holes in such a system one would have, as seen by an outside observer, arbitrarily compressed length in the radial direction, and arbitrarily slow passage of time as one approached the horizon. We make some additional assumptions on the horizon in order to ensure a physically reasonable situation. These are:

1. The volume element $\sqrt{|\det g_{ij}|}$ is a smooth function and is nonzero at the horizon (the horizon is *regular*: that is, it is not locally distinguishable from other points in spacetime, which is physically reasonable, since only the center of the black hole should be singular).

2. The electromagnetic field strength, $F_{ij}F^{ij} = -2|\phi'|^2 AT^2$ (from the Faraday tensor F_{ij}) is bounded near the horizon (again, the horizon's singularity is a coordinate artifact).

3. $A(r)$ obeys a power law. That is, there exist constants C and s, both positive, such that outside the horizon (i.e. for $r > \rho$) the following is

true:

$$A(r) = C(r-\rho)^s + \mathcal{O}((r-\rho)^{s+1}) \tag{4.28}$$

The regularity condition (1) is equivalent to the condition that T^2A and its inverse $(T^2A)^{-1}$ are both smooth functions of r on the interval $[\rho, \infty)$, since $\sqrt{|detg_{ij}|} = r^2 A^{-\frac{1}{2}} T^{-1}$, and clearly r^2 is smooth here, so if both of these are smooth, the volume element is as well, and nonzero on the horizon since T^2A must be defined at $r = \rho$. Because of the boundedness of this function, the condition on the strength of the electromagnetic field, (2) simply becomes the boundedness of $|\phi'(r)|$ for $r \in (\rho, \rho + \epsilon)$.

The first two conditions describe physically reasonable black hole solutions: the principle is that the only singularity should be that at the center of the black hole. There is a coordinate singularity at the horizon in polar coordinates, as can be seen by the fact that the functions A and T appearing in the metric in this system are not well behaved. However, physically significant scalar quantities are well behaved near the horizon. The regularity condition (1) that the volume element should be nonzero means that objects passing the horizon would not be "crushed", and the condition on the finite strength of the electromagnetic field means that there are only physically reasonable forces acting there.

4.2.2 Main Theorem

In this section, we develop the results leading up to and including the main nonexistence theorem of this chapter, which states that the only black hole solutions to the EDM system (4.22)-(4.26) is the non-extreme Reissner-Nordström solution. There are two cases, to which we will devote a lemma each - the two cases are separated for reasons related to the form of the proof. The first case is for the power-law $s < 2$: this contains the Reissner-Nordström case, with $s = 1$, and here we show that there are no solutions with the Dirac spinors not identically zero (where they are identically zero, we have the R-N solution). We begin this with a technical lemma, showing that for such a solution, the spinors are finite and bounded away from zero near the horizon. More precisely:

Lemma 1. *If the power-law in condition 3 has power $s < 2$ and there is a black hole solution (α, β, A, T) to the EDM system (4.22)-(4.26) for which the spinors α and β are not everywhere zero, then $(\alpha^2 + \beta^2)$ is bounded from above and below near $r = \rho$: that is, $\exists \epsilon > 0$ and $\exists c > 0$ for which*

$$c \leq \alpha^2 + \beta^2 \leq c^{-1} \text{ when } \rho < r < \rho + \epsilon \tag{4.29}$$

Proof. If we take

$$\sqrt{A}\frac{d}{dr}(\alpha^2 + \beta^2) = 2\sqrt{A}(\alpha\alpha' + \beta\beta')$$

then by the Dirac equations 4.25 and 4.26, we can write this as:

$$\sqrt{A}\frac{d}{dr}(\alpha^2 + \beta^2) = 2\begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}\pm\frac{2j+1}{2r} & -m \\ -m & \mp\frac{2j+1}{2r}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

where the terms involving $((\omega - e\phi)T)\alpha\beta$ have canceled. We can get an upper bound on the magnitude of this last, using the operator of the matrix in the middle, so that

$$\sqrt{A}\frac{d}{dr}(\alpha^2 + \beta^2) \leq \left(4m^2 + \frac{(2j+1)^2}{r^2}\right)^2(\alpha^2 + \beta^2) \tag{4.30}$$

But now, we have found an ODE for $(\alpha^2 + \beta^2)(r)$, and since solutions to this ODE are uniquely determined (locally) by their values at any point, we know that a nontrivial such solution will be nonzero everywhere on some interval $\rho < r < \rho + \epsilon$. Writing the bound above as:

$$\frac{\frac{d}{dr}(\alpha^2 + \beta^2)}{(\alpha^2 + \beta^2)} \leq \frac{1}{\sqrt{A}}\Big(4m^2 + \frac{(2j+1)^2}{r^2}\Big)^2$$

we have the left hand side to be an exact differential (of $\log(\alpha^2 + \beta^2)$), so we can integrate from $r = \rho$ to $r = \rho + \epsilon$ and obtain that

$$\begin{aligned} &\big|\log((\alpha^2 + \beta^2)(\rho + \epsilon)) - \log((\alpha^2 + \beta^2)(r))\big| \\ &\qquad\qquad \leq \int_{\varrho = r}^{\rho + \epsilon} \frac{1}{\sqrt{A(\varrho)}}\Big(4m^2 + \frac{(2j+1)^2}{\varrho^2}\Big)^2 d\varrho \end{aligned} \tag{4.31}$$

Now recalling that $A(r)$ satisfies the power law (3), with (in this case) $s < 2$, we see that $A(r)$ as $r \mapsto \rho^+$, the term $\frac{1}{\sqrt{A}}$ does not grow too rapidly, and so we can take the limit as $\varrho \mapsto r$, since the function is integrable. This yields some finite value, and hence near the horizon the spinors are bounded, as required for the statement of the lemma. □

This lemma will be important for the proof of the case of the main theorem where $s < 2$, since it will allow us to show that the electromagnetic field strength near the horizon is infinite if $s \geq 1$ on the one hand (violating condition 2) or to obtain a contradiction from the boundedness above if $s < 1$ on the other hand. For the case where $s > 2$, we will need to use a different method of proof, since this lemma no longer applies. We shall use two more technical lemmas, of which the following is the first:

Lemma 2. *If $s \geq 2$, then*

$$\lim_{r \mapsto \rho^+} (r - \rho)^{-\frac{s}{2}} (\alpha^2 + \beta^2)(r) = 0 \tag{4.32}$$

Proof. Consider the Maxwell equation 4.24 and write it as

$$\phi'' = -\frac{1}{A}\frac{(2j+1)e}{r^2}(\alpha^2+\beta^2) - \frac{(r^2\sqrt{A}T)'}{r^2\sqrt{A}T}\phi' \tag{4.33}$$

(where of course all primed derivatives are radial derivatives). Now by the regularity condition 2, $|\phi'|$ is bounded near the horizon, while from the regularity condition we know that $r^2\sqrt{A}T$ is smooth (hence its derivative exists and is bounded) and nonzero near the horizon, so that the second term in the expression for ϕ'' is also smooth, and thus we must have that the first term is integrable, or:

$$\int_\rho^{\rho+\epsilon} \frac{1}{A}(\alpha^2+\beta^2) < \infty \tag{4.34}$$

(since the other part is bounded on both sides near the horizon and does not affect integrability).

Now if we consider the function in the limit on the left hand side of 4.32, and take its derivative, we find:

$$\frac{d}{dr}\big((r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2)\big) = -\frac{s}{2}(r-\rho)^{-\frac{s}{2}-1}(\alpha^2+\beta^2) + (r-\rho)^{-\frac{s}{2}}\frac{d}{dr}(\alpha^2+\beta^2)$$

into which we can substitute the bound we found in 4.30 for $\frac{d}{dr}(\alpha^2+\beta^2)$, to get the bound

$$\begin{aligned}&\Big|\frac{d}{dr}\big(r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2)()\Big|\\ &\leq -\frac{s}{2}(r-\rho)^{-\frac{s}{2}-1}(\alpha^2+\beta^2) + \frac{1}{\sqrt{A}}\Big(4m^2+\frac{(2j+1)^2}{r^2}\Big)^2(\alpha^2+\beta^2)(r-\rho)^{-\frac{s}{2}}\end{aligned} \tag{4.35}$$

But now if we notice that both of the terms of this expression are integrable, since in each case the function is bounded by a function of the same order as 4.34 and thus converges: in each case, this is so because $s \geq 2$. But since each of these is integrable, so is the left side - since the magnitude of the derivative of $(r-\rho)^{-\frac{s}{2}}(\alpha^2+\beta^2)$ is integrable, the function itself must have some limit as one approaches the horizon - but since by the condition 3, we have $A = \mathcal{O}(r-\rho)^s$, if the limit were anything other than zero, 4.34 could not be integrable since near the horizon it would behave as a function of order $\mathcal{O}(r-\rho)^{-\frac{s}{2}}$, which would, since $s \geq 2$, diverge. Thus, the lemma must hold. □

The last technical lemma we will need is the following:

Lemma 3. *If $s > 0$ then $|\phi'|$ has a finite, nonzero limit as $r \mapsto \rho$:*

$$\lim_{r \mapsto \rho^+} |\phi'| = \frac{1}{\rho} \lim_{r \mapsto \rho^+} A^{-\frac{1}{2}} T^{-1} > 0$$

Proof. Consider the equation 4.22. Since we have $s \geq 2$, by the power-law for A, the left side of this equation approaches zero as $r \mapsto \rho^+$, so we must have that:

$$\lim_{r \mapsto \rho^+} (1 - A - 2(2j+1)(\omega - e\phi)T^2(\alpha^2+\beta^2) - r^2 A T^2 |\phi'(r)|^2)$$

Now clearly since we have regularity of the horizon is that the term $2(2j+1)(\omega - e\phi)T^2(\alpha^2+\beta^2)$ should have some finite limit as one approaches the horizon, and the result will follow. But now note that if $|(\omega - e\phi)T|$ is bounded, then the previous lemma yields the desired result. So suppose otherwise - if $|(\omega - e\phi)T|$ is not bounded near $r = \rho$, then consider the differential equation for AT^2 arising from the Einstein equations 4.22 and

4.23:

$$r\frac{d}{dr}(AT^2) = -4(2j+1)(\omega - e\phi)T^4(\alpha^2+\beta^2) \tag{4.36}$$

$$\pm\, 2\frac{(2j+1)^2}{r}T^3\alpha\beta + 2(2j+1)mT^3(\alpha^2-\beta^2) \tag{4.37}$$

Using this to estimate bounds on the magnitude of the derivative in question, this gives us the estimate

$$\left|r\frac{d}{dr}AT^2\right| \geq T^3(\alpha^2+\beta^2)\Big(4(2j+1)|(\omega-e\phi)T| - 2\frac{(2j+1)^2}{r} - 2(2j+1)m\Big)$$

Now the regularity condition (1) on the horizon means that the left side of this must be bounded, and since our assumption was that $|(\omega - e\phi)T|$ is unbounded and the rest of the bracketed terms are all bounded, we must have that the product $T^3(\alpha^2+\beta^2)|(\omega-e\phi)T|$ is bounded, but since T grows without bound as we approach the horizon, this means that $\lim_{r\mapsto\rho^+}(\omega - e\phi)T^2(\alpha^2+\beta^2) = 0$, which we have already seen will yield the result desired. □

With these technical lemmas in hand, we can proceed to the proof of the main nonexistence theorem we wish to demonstrate for the case of the static, spherically symmetric EDM system, which is the following:

Theorem 1. *The only black hole solutions for the Einstein-Dirac-Maxwell system (4.22)-(4.26) for which the horizon satisfies regularity conditions 1, 2 and 3 are the non-extreme Reissner-Nordström solution (where $\alpha = \beta = 0$, hence the Dirac field vanishes) and the case where $s = 2$ and in this case,* *near $r = \rho$ the following expansions hold:*

$$
\begin{aligned}
A(r) &= A_0(r-\rho)^2 + \mathcal{O}((r-\rho)^3) \\
T(r) &= T_0(r-\rho)^{-1} + \mathcal{O}((r-\rho)^0) \\
\phi(r) &= \frac{\omega}{e} + \phi_0(r-\rho) + \mathcal{O}((r-\rho)^2) \\
\alpha(r) &= \alpha_0(r-\rho)^\kappa + \mathcal{O}((r-\rho)^{\kappa+1}) \\
\beta(r) &= \beta_0(r-\rho)^\kappa + \mathcal{O}((r-\rho)^{\kappa+1})
\end{aligned} \tag{4.38}
$$

where A_0 and T_0 are positive real, and ϕ_0, α_0 and β_0 are real. The power κ in the expansion for α and β must satisfy

$$\frac{1}{2} < \kappa = \frac{1}{A_0}\sqrt{m^2 - e^2\phi_0^2 T_0^2 + \big(\frac{2j+1}{2\rho}\big)^2} \tag{4.39}$$

and α_0 and β_0 are related by:

$$\alpha_0\big(\sqrt{A_0}\kappa \pm \frac{2j+1}{2\rho}\big) = -\beta_0(m - e\phi_0 T_0) \tag{4.40}$$

where $c = \pm$.

Proof. There are three cases here, split by the special case where $s = 2$.

Case 1: $(0 < s < 2)$ This case includes the Reissner-Nordström solutions (where there is no Dirac field - or in other words, the spinors $(\alpha, \beta) \cong (0, 0)$) - to show that there are no others, we proceed with a proof by contradiction. Suppose that there is a black hole solution to the system (4.22)-(4.26) in which the spinors do not identically vanish. Then we can use the lemma 1, and we have that (α, β) are bounded near the horizon at $r = \rho$. Considering the DE (4.36) for AT^2, we again have the left side smooth because of the regularity condition on the horizon, and thus the right side smooth as well. The now as $r \mapsto \rho$, $T \mapsto \infty$, and so the dominant term of the right is the term $-4(2j+1)(\omega-e\phi)T^4(\alpha^2+\beta^2)$, so this must be smooth, hence, since $(\alpha^2+\beta^2)>0$ and $T\mapsto\infty$, we must have that $(\omega-e\phi)\mapsto 0$.

Now take Maxwell's equation in the form previously found in (4.33) and note that by the regularity conditions, the term for ϕ' is smooth. If $s\geq 1$, however, the singularity for A^{-1} in the other term is not integrable, hence $|\phi'|\mapsto\infty$ as $r\mapsto\rho^+$, which contradicts the regularity of the electromagnetic field on the horizon (regularity condition (2)). On the other hand, if $s<1$, then we can integrate (4.33) to get a form for ϕ' around the horizon, since the term A^{-1} has an integrable singularity. The result can again be integrated, and, obtaining the constant of integration at this stage from the fact that $\lim_{r\mapsto\rho^+}(\omega-e\phi(r))=0$, giving the formula for the expansion of ϕ about the horizon:

$$\phi(r)=c_1(r-\rho)^{-s+2}+c_2(r-\rho)+\frac{\omega}{e}+\mathcal{O}((r-\rho)^{-s+3})$$

Upon substituting this form for ϕ into the Einstein equation for A, namely (4.22), we see that since the right side of (4.22) is bounded as $r\mapsto\rho^+$ since A and $r^2AT^2|\phi'|^2$ are bounded and $(\omega-e\phi)=\mathcal{O}(r-\rho)$, while $T^2(\alpha^2+\beta^2)$ is of order $(r-\rho)^{-s}$, with s strictly less than 1. But the left side is not bounded, being of order $(r-\rho)^{s-1}$, which is impossible. This yields a contradiction, so in fact there can be no such solutions.

Case 2: $(s>2)$ By Lemma 3, we have a Taylor series expansion around $r=\rho$ for $(\omega-e\phi)$ of:

$$(\omega-e\phi)(r)=c+d(r-\rho)+R(r-\rho)$$

where $0 \neq |d| = \frac{e}{\rho} \lim_{r \mapsto \rho^+} \frac{1}{\sqrt{AT}}$.

So near the horizon, the term $(\omega - e\phi)T$, which appears in both the Dirac equations (4.25) and (4.26), diverges monotonically. Now by the Dirac equations for the spinor components α and β, namely 4.25 and 4.26, we can see that since the derivative for each component contains this term multiplied by the other component, we have the spinor (α, β) spinning about the origin at an increasing rate as one approaches the horizon. In [FSY3] it is shown that in general if we have an ODE

$$\Phi'(x) = \left[a(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + b(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \Phi(x)$$

with smooth coefficients a , b , c, and with $\frac{b}{a}$ and $\frac{c}{a}$ smooth and monotone near $x = 0$, and $b^2 + c^2 < a^2$, then $|\Phi|^2$ is bounded above and below near $x = 0$. (The proof of this involves some relatively straightforward analysis involving the functional given by the Hermitian matrix

$$A(x) = \begin{pmatrix} 1 + \frac{b}{a} & -\frac{c}{a} \\ -\frac{c}{a} & 1 - \frac{b}{a} \end{pmatrix}$$

and its norm)The Dirac equation in the form (4.15) is now such a system, with $x = (r - \rho)$, so we would have (α, β) bounded away from zero, which is in contradiction with lemma 2, and so we have a contradiction.

Case 3: $(s = 2)$ In this case, which must include the Reissner-Nordström case, we have from the power-law condition on A that the first two Taylor expansions in 4.38 must hold - this is part of the theorem proved. Now again by results shown in [FSY3] we have that $(\omega - e\phi)T$ cannot diverge monotonically near the horizon. But by Lemma 3 implies that it has a Taylor expansion around $r = \rho$ with nontrivial linear coefficient $d = \lim_{r \mapsto \rho^+} \frac{e}{\rho} \frac{1}{\sqrt{AT}}$. So the constant term in the expansion of the bare $(\omega - e\phi)$ must be zero because of the expansion for T, and we have

$$\lim_{r \mapsto \rho^+} (\omega - e\phi) T = \lambda$$

with

$$|\lambda| = \frac{e}{\rho} \lim_{r \mapsto \rho^+} \frac{(r-\rho)^{-1}}{\sqrt{A}}$$

(where we have used, from Lemma 3, the form for the derivative of ϕ to find this form).

Now we can rewrite the Dirac equations 4.25 and 4.26 in the transformed variable $u(r) = -r - \rho \ln(r - \rho)$, which approaches infinity as one approaches the horizon. The qualitative theory of ODEs, and in particular the linear stable manifold theorem, describes the asymptotic behaviour of solutions to such an equation, and we can thus determine that α and β satisfy the required power law from 4.38, while the constraint that $\kappa > \frac{1}{2}$ is the result of the fact that $\lim (r-\rho)^{-1}(\alpha^2 + \beta^2)(r) = 0$ (the result of lemma 2 in the case $s = 2$), since α and β were of higher order than $\frac{1}{2}$ in $(r - \rho)$, this would diverge.

The remaining constraint relating the spinor coefficients α_0 and β_0 is derived by substituting the taylor expansions we have thus obtained into the Dirac equations.

$\square$

Chapter 5

Case II: Time Periodic Solutions of Dirac Equation in Axisymmetric Black Hole Geometry

In the first part, we dealt with an existence theorem for the Einstein-Dirac system in the case of spherical symmetry, and in doing so developed some of the analytic tools for examining such systems. In this part, on nonexistence theorems for black hole solutions, we are considering as well some slightly different situations. In chapter 4, we added electromagnetism and dealt with a fully coupled Einstein-Dirac-Maxwell system, still in a spherically symmetric situation. In this chapter, we shall relax full coupling and consider the Dirac equation acting on a fixed background, but this will enable us to relax the symmetry requirement, allowing perfect spherical symmetry of the black hole spacetime to be deformed to an axisymmetric geometry. The Dirac field will still be coupled to gravity and to electromagnetism, but the coupling of the metric and the Faraday tensor to the Dirac field is ignored.

This is a reasonable approximation for the case where the mass and charge of the Dirac particle is small compared to that of the black hole, as would typically be the case in physical situations. The advantage of this approach is that it allows us to consider a far more general black hole geometry. In particular, we shall begin with a consideration of the Kerr-Newman black hole geometry, which is the most general geometry involving the coupling of gravity to electromagnetism. On this background, we will show that there do not exists time periodic solutions of the Dirac equation, which we do by first decomposing such solutions into Fourier series and considering the various components as static solutions. The proof that there do not exist static such solutions resembles that for the Reissner-Nordström background.

5.1 Kerr-Newman Geometry

The Kerr-Newman geometry is the most general black-hole solution for the Einstein-Maxwell equation, and forms the background which we shall consider for the solutions of the Dirac equation in this situation. The Kerr-Newman geometry is parametrized by the mass of the black hole (as measured from infinity), its angular momentum (also measured from infinity) and its charge. That is, the Kerr-Newman solution is characterized by the parameters (a, Q, M), where a is the angular momentum per unit mass $\frac{S}{M}$, Q is the charge, and M is the ADM mass (see for instance box 33.2 of [MTW], pp878-883). There is a horizon only if the mass is sufficiently high to overcome the repulsive effects of angular momentum and the associated frame dragging, as well as the charge of the hole. This occurs if and only if $M^2 \geq Q^2 + a^2$ - for smaller mass, one obtains a "naked singularity", namely a singularity of curvature without the presence of a horizon, which is presumed to be a non-physical phenomenon. The limiting case with $M^2 = Q^2 + a^2$ is the so-called "extreme Kerr-Newman geometry", and the result we shall show does not apply to such a case.

The various degenerate cases where the angular momentum, the charge, or both, vanish yield, respectively, the Reissner-Nordström, the Kerr, and the Schwarzchild solutions, which together are all the stationary geometries for the Einstein-Maxwell equation. The result we shall deal with here shows that there can be no time-periodic solutions for the Dirac equation on this geometry (note that this result also generalizes the condition in the previous result, pertaining only to static solutions). This shows, in this more general class of geometries, albeit for only partial coupling, a result similar to that of chapter 4 holds. To consider solutions of the Dirac equation on this background, we must make use of several coordinate systems, because it is necessary to extend the solutions across the event horizon (though this can only be done in the distributional sense). The generalized Dirac equation is discontinuous at the event and Cauchy horizons of the KN geometry, but may still be analyzed by techniques similar to those used in the spherically symmetric, completely coupled case, involving the asymptotic behaviour of certain spinors as one approaches the event horizon. The main difference is that one must derive conditions at both types of horizon in order to extend the solutions obtained outside the black hole into the interior, since there may be more than one "asymptotic end", or asymptotically flat portion of spacetime, in the Kerr-Newman geometry.

5.1.1 Form of Dirac Equation in KN Geometry

A convenient set of coordinates on the KN background are the Boyer-Lindquist coordinates (t, r, θ, ϕ) in which the metric has the form

$$\begin{aligned} ds^2 =& \frac{\delta}{U}(dt - a\sin^2\theta d\phi)^2 - U\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) \\ &- \frac{\sin^2\theta}{U}(adt - (r^2 + a^2)d\phi)^2 \end{aligned} \tag{5.1}$$

with

$$U(r,\theta) = r^2 + a^2\cos^2\theta \ , \ \Delta(r) = r^2 - 2Mr + a^2 + Q^2 \tag{5.2}$$

(where e.g. [MTW] uses ρ^2, we use U in conformity with [FKSY1]) in these coordinates, we also have that the potential A for the electromagnetic field is

$$\mathbf{A} = A^j dx^2 = -\frac{Qr}{U}(dt - a\sin^2\theta d\phi) \tag{5.3}$$

Now in these coordinates, we observe that there are several cases where the metric becomes singular. When $r = 0$, which is at the singularity itself, the metric blows up, as does the curvature tensor, resulting from the presence of the U in the form of the metric. But the metric also blows up at the roots of Δ. Now in a non-rotating, non-charged black hole, the equivalent of Δ is a linear function with only one root (at the horizon), but here there are two. These correspond to the *event horizon* and the *Cauchy horizon*, and occur at, respectively:

$$r_1 = M + \sqrt{M^2 - a^2 - Q^2}$$

and

$$r_0 = M - \sqrt{M^2 - a^2 - Q^2}$$

In these coordinates, we construct a frame of vectors at each point from which we will construct the Dirac matrices. This will be a so-called Newman-Penrose frame, which is a way of choosing a basis for the tangent space at each point in a way which simplifies certain calculations. The frame produced is a null frame - that is, it consists of null vectors. The Newman-Penrose method is motivated (cf. [Wald] pp52, 372-373) by considering a basis for the space of Pauli 2-spinors at each point, say o^A and ι^A, having $o_A \iota^A = 1$. One constructs the frame by taking two null vectors as the vectors corresponding to the product of these two basis spinors with themselves:

$$l^{AA'} = \iota^A \overline{\iota}^{A'}$$

and

$$n^{AA'} = o^A \overline{o}^{A'}$$

together with two vectors obtained by the other two possible multiplications of the basis spinors with their complex conjugates, namely the complex (and mutually conjugate) vectors:

$$m^{AA'} = \iota^A \overline{o}^{A'}$$

and

$$\overline{m}^{AA'} = o^A \overline{\iota}^{A'}$$

(recall that ordinary vectors can be formed as the product of spinors and complex-conjugate spinors, so that we really have here a null tetrad of vectors $(l^a, n^a, m^a, \overline{m}^a)$). Such a null frame has the property that $l^a n_a = 1$, $m^a \overline{m}_a =$

-1 and all other inner products between its elements vanish. The complex vector m may be considered as a complex linear combination of real vectors corresponding to its real and imaginary parts, say $m^a = \frac{1}{\sqrt{2}}(x^a + iy^a)$ where x and y are unit spacelike vectors orthogonal to l and n.

In order to obtain a useful form for the Dirac operator on the KN background, we make use of a frame (the *symmetric frame*) of this kind, which is expressed in Boyer-Lindquist coordinates as:

$$\begin{aligned}
l &= \frac{1}{\sqrt{2U|\Delta|}}\Big((r^2+a^2)\frac{\partial}{\partial t} + \Delta\frac{\partial}{\partial r} + a\frac{\partial}{\partial\phi}\Big) \\
n &= \frac{\epsilon(\Delta)}{\sqrt{2U|\Delta|}}\Big((r^2+a^2)\frac{\partial}{\partial t} - \Delta\frac{\partial}{\partial r} + a\frac{\partial}{\partial\phi}\Big) \\
m &= \frac{1}{\sqrt{2U}}\Big(ia\sin\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big) \\
\overline{m} &= \frac{1}{\sqrt{2U}}\Big(-ia\sin\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big)
\end{aligned} \tag{5.4}$$

In this expression, we have used the step function $\epsilon(x)$ which is 1 for nonnegative x and 0 for negative x. Applied to Δ, which is negative between the Cauchy horizon and the event horizon, this makes the frame degenerate there (with $n = 0$).

From this Newman-Penrose null frame, we can construct a real orthonormal frame by taking linear combinations of the symmetric frame vectors:

$$\begin{aligned}
u_0 &= \frac{\epsilon(\Delta)}{\sqrt{2}}(l+n) \\
u_1 &= \frac{1}{\sqrt{2}}(l-n) \\
u_2 &= \frac{1}{\sqrt{2}}(m+\overline{m}) \\
u_3 &= \frac{1}{\sqrt{2}i}(m-\overline{m})
\end{aligned} \tag{5.5}$$

We note again that this frame is degenerate between the Cauchy and event horizons. It is this frame which we shall use in representing vectors, rather than the coordinate frame, since this makes our calculation simpler.

Now if we consider the Dirac equation in this background, we first remark that, while the form (A.3) for the E matrices remains the same as before, the spinor connection D giving the spin derivative for the Dirac operator $(iG^j D_j)$ now has a slightly different form due to the presence of an electromagnetic field with potential A, which influences the Dirac particles if they are charged. The spin derivative D is thus of the form

$$D_j = \frac{\partial}{\partial_j} - iE_j - ieA_j \tag{5.6}$$

so that the B matrix has the form

$$B = G^j(E_j + eA_j)$$

The analysis of this is very similar to the case in part I, except for the presence of the electromagnetic potential term. One can, by similar means to those used there, find a form for the B matrix which involves only partial derivatives, namely:

$$B = \frac{i}{2\sqrt{|g|}}\partial_j(\sqrt{|g|}u_a^j)\gamma^a - \frac{i}{4}\epsilon^{jmnp}\eta^{ab}u_{am}(\partial_j u_{bn})u_{cn}\gamma^5\gamma^c + eA_j u_a^j\gamma^a \tag{5.7}$$

Now from this, the explicit form of the Dirac operator as

$$G = iG^j\frac{\partial}{\partial x^j} + B(x)$$

and the explicit form for the γ matrices and the u-basis for vectors, it is straightforward to compute the Dirac operator directly, though the terms are somewhat complicated.

The matrix for the Dirac operator has a fairly symmetric form, involving the following terms:

$$\beta_{\pm} = \frac{1}{\sqrt{U}}\Big(i\partial_\theta + \frac{i}{2}\cot\theta + \frac{a\sin\theta}{2U}(r - ia\cos\theta)\Big) \pm \frac{1}{\sqrt{U}}\Big(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\phi\Big)$$

and

$$\overline{\beta}_{\pm} = \frac{1}{\sqrt{U}}\Big(i\partial_\theta + \frac{i}{2}\cot\theta - \frac{a\sin\theta}{2U}(r + ia\cos\theta)\Big) \pm \frac{1}{\sqrt{U}}\Big(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\phi\Big)$$

and

$$\begin{aligned}\alpha_{\pm} = &- \frac{\epsilon(\Delta)}{\sqrt{U|\Delta|}}\Big(i(r^2 + a^2)\partial_t + ia\partial_\phi + eQr\Big) \\ &\pm \sqrt{\frac{|\Delta|}{U}}\Big(i\partial_r + i\frac{r - M}{2\Delta} + \frac{i}{2U}(r - ia\cos\theta)\Big)\end{aligned}$$

and

$$\begin{aligned}\overline{\alpha}_{\pm} = &- \frac{\epsilon(\Delta)}{\sqrt{U|\Delta|}}\Big(i(r^2 + a^2)\partial_t + ia\partial_\phi + eQr\Big) \\ &\pm \sqrt{\frac{|\Delta|}{U}}\Big(i\partial_r + i\frac{r - M}{2\Delta} + \frac{i}{2U}(r + ia\cos\theta)\Big)\end{aligned}$$

With these terms, the Dirac operator may be written as:

$$G = \begin{pmatrix} 0 & 0 & \alpha_+ & \beta_+ \\ 0 & 0 & \beta_- & \epsilon(\Delta)\alpha_- \\ \epsilon(\Delta)\overline{\alpha}_- & -\overline{\beta}_+ & 0 & 0 \\ -\overline{\beta}_- & \overline{\alpha}_+ & 0 & 0 \end{pmatrix} \tag{5.8}$$

The Dirac Equation is then $(G - m)\Psi = 0$ for this G, acting on Dirac 4-spinor fields Ψ.

It was first shown by Chandrasekhar in 1976 that this Dirac equation can be separated completely into ODE's in the Kerr background, and later extended to the KN background by Page and by Toop. Although we will not enter into great detail, we will now discuss this separation. It proceeds by making a gauge transformation of the wave function into a new form, where the Dirac operator is transformed into a form which separates into a sum of two operators with dependence upon only, respectively, radial and angular coordinates.

If we consider the matrices

$$S = |\Delta|^{\frac{1}{4}} \begin{pmatrix} (r - ia\cos\theta)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & (r - ia\cos\theta)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & (r + ia\cos\theta)^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & (r + ia\cos\theta)^{\frac{1}{2}} \end{pmatrix} \tag{5.9}$$

and

$$\Gamma = -i \begin{pmatrix} (r + ia\cos\theta) & 0 & 0 & 0 \\ 0 & -(r + ia\cos\theta) & 0 & 0 \\ 0 & 0 & -(r - ia\cos\theta) & 0 \\ 0 & 0 & 0 & (r - ia\cos\theta) \end{pmatrix} \tag{5.10}$$

Then one can consider a gauge transformation of the wave function Ψ by S, namely $\hat{\Psi} = S\Psi$, and consider the effect of this on the property that Ψ should satisfy the Dirac equation. This becomes:

$$\Gamma S(G - m)S^{-1}\hat{\Psi} = 0 \tag{5.11}$$

so the Dirac operator is now represented as $\Gamma S(G - m)S^{-1}$, which, when calculated, turns out to be the sum of the following two operators:

$$\mathcal{R} = \begin{pmatrix} imr & 0 & \sqrt{|\Delta|}\mathcal{D}_+ & 0 \\ 0 & -imr & 0 & \epsilon(\Delta)\sqrt{|\Delta|}\mathcal{D}_- \\ \epsilon(\Delta)\sqrt{|\Delta|}\mathcal{D}_- & 0 & -imr & 0 \\ 0 & \sqrt{|\Delta|}\mathcal{D}_+ & 0 & imr \end{pmatrix}$$

and

$$\mathcal{A} = \begin{pmatrix} -am\cos\theta & 0 & 0 & \mathcal{L}_+ \\ 0 & am\cos\theta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am\cos\theta & 0 \\ -\mathcal{L}_- & 0 & 0 & am\cos\theta \end{pmatrix}$$

where we have

$$\mathcal{D}_\pm = \partial_r \mp \frac{1}{\Delta}\Big((r^2+a^2)\partial_t + a\partial_\phi - ieQr\Big)$$

and

$$\mathcal{L}_\pm = \partial_\theta + \frac{1}{2}\cot\theta \mp i\Big(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\phi\Big)$$

Now since $\mathcal{R}$ is dependent only on radial variables and $\mathcal{A}$ only on angular ones, and the transformed Dirac operator on $\hat{\Psi}$ is their sum, we would expect to try to find an ansatz for $\hat{\Psi}$ which reflects this separation by allowing the system to decompose into two independent systems. These systems will be for 2-spinors, involving 2×2 matrices as operators, so we need an ansatz involving two 2-component functions, one dependent upon r and the other upon θ. If we first remove the time dependence to get the form

$$\hat{\Psi}(t,r,\theta,\phi) = e^{-i\omega t - ik\phi}\hat{\Phi}(r,\theta) \tag{5.12}$$

for real energy ω and half-odd-integral spin k. We then say that if our unknown 2-component radial function is $X_\pm(r)$ and the angular one is $Y_\pm(\theta)$, we can construct the whole form for $\hat{\Phi}$ from these as:

$$\hat{\Phi}(r,\theta) = \begin{pmatrix} X_-(r)Y_-(\theta) \\ X_+(r)Y_+(\theta) \\ X_+(r)Y_-(\theta) \\ X_-(r)Y_+(\theta) \end{pmatrix} \tag{5.13}$$

The form for Ψ realized from these expressions gives, when we apply the Dirac operator $\mathcal{R}+\mathcal{A}$ to the gauge-transformed wavefunction, gives, clearly, $(\mathcal{R}+\mathcal{A})\hat{\Psi}=0$, but in fact we have more, namely that since $\mathcal{R}$ acts purely on the radial component and $\mathcal{A}$ acts purely on the angular component, this can only happen if $\hat{\Psi}$ is a simultaneous eigenstate of the two operators, so that

$$\mathcal{R}\hat{\Psi}=-\mathcal{A}\hat{\Psi}=\lambda\hat{\Psi}$$

From this, we find that the Dirac equation decouples. From the equation $\mathcal{R}\hat{\Psi}=\lambda\hat{\Psi}$ we obtain:

$$\begin{pmatrix} \sqrt{|\Delta|}\mathcal{D}_+ & imr-\lambda \\ -imr-\lambda & \epsilon\Delta\sqrt{|\Delta|}\mathcal{D}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0 \tag{5.14}$$

while from the equation $\mathcal{A}\hat{\Psi}=-\lambda\Psi$ we get:

$$\begin{pmatrix} \mathcal{L}_+ & -am\cos\theta+\lambda \\ am\cos\theta+\lambda & -\mathcal{L}_- \end{pmatrix} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix} = 0 \tag{5.15}$$

Furthermore, the operators $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ can be simplified, since each is applied to a purely radial or purely angular term, respectively, so the other derivatives involved in each operator vanish, leaving the forms:

$$\mathcal{D}_\pm=\partial_r\pm\frac{i}{\Delta}(\omega(r^2+a^2)+ka+eQr)$$

and

$$\mathcal{L}_\pm=\partial_\theta+\frac{1}{2}\cot\theta\mp(a\omega\sin\theta+k\csc\theta)$$

5.2 Nonexistence Theorem

5.2.1 Matching Conditions for Spinors Across Horizon

Having obtained the ODE forms (5.14) and (5.15) above for the Dirac equation, we now note that there is more to be done before we can proceed to analyze the solutions to this system. In particular, we observe some difficulties with the radial equation, which not only posesses a discontinuity due to the step function $\epsilon(\Delta)$, but also, because of the presence of the term Δ^{-1} in the expression for $\mathcal{D}$, is singular at the roots of Δ, namely $r = r_0$ and $r = r_1$. The existence of these poles in the coefficients of the equation means that the solutions - the wavefunctions representing the Dirac field - may have discontinuities at these values of r, may in fact have singular behaviour as r approaches them. The peculiar causal structure of spacetime at these, the event and Cauchy horizons, also makes treatment of a wavefunction defined across them somewhat problematic. Consequently, it is necessary to develop *matching conditions* across the horizon, which will give conditions on the relationship between the solutions to the Dirac equation inside and outside both type of horizon.

The maximal analytic extension of the Kerr-Newman geometry (part of whose conformal diagram is shown in figure 5.2.1) consists of an infinite number of copies of three types of regions, which are designated I, M, and O because they represent regions Inside the Cauchy Horizon, in the Middle (between the Cauchy and Event Horizons) and Outside the Event Horizon. That is, in our Boyer-Lindquist coordinates, the region I represents the part of the spacetime with $r < r_0$, region M is where $r_0 < r < r_1$ and O is where $r_1 < r$. This is similar to the situation with the Kerr solution as in, for instance, ([Hawk] p165). Thus, we are considering conditions which allow us to extend solutions to our equation, or find constraints satisfied by solutions extending one given on some part of the maximum analytic extension.

To derive such conditions, we must obtain the Dirac operator in a form

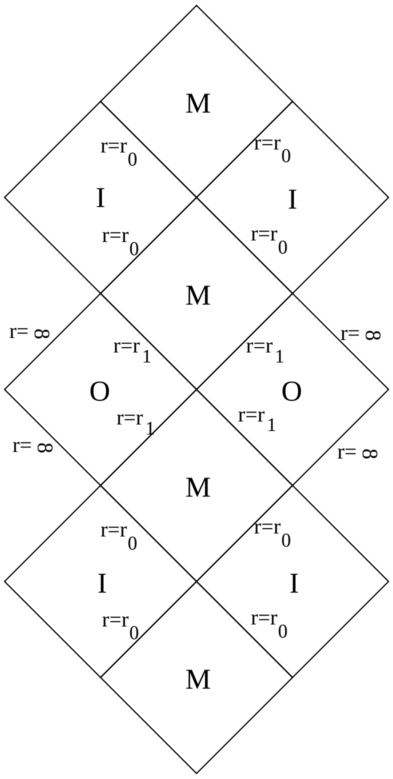

Figure 5.1: Part of the Penrose Diagram for the Maximal Analytic Extension of Kerr-Newman Geometry

which extends across the horizons - this is done by expressing the operator in Kerr coordinates, which are not singular at the horizons. Actually, there are two types of Kerr coordinates, one for each horizon which we wish to cross. In each case, we will construct the wavefunction Ψ as the sum of two distinct solutions, one for each of the two regions bordering the horizon in question. This requires the use of a step function (in particular, the Heaviside function Θ) to multiply by the two solutions, since the wavefunction as a whole need not be smooth, and so neither solution need necessarily be extendible smoothly across the horizon. Derivatives of these yield Dirac delta functions, and the necessity of defining solutions of the Dirac equation in the distributional sense.

The Kerr coordinates resolve some of the difficulties of the Boyer-Lindquist coordinates for our purposes - we relate them infinitesimally (the relation of coordinate values themselves can be recovered - up to an irrelevant constant - by integration) in order to make easier the transformations of the Dirac matrices, which transform, we recall, as local coordinate basis vectors. There are two possible Kerr coordinate patches of interest, one crossing the Event horizon, which we designate with a $+$ index, and the other, indicated with a $-$, which we use to derive the matching across the Cauchy horizon. The coordinates r and θ remain unchanged, but the Kerr coordinates have new variables $u_\pm$ and $\phi_\pm$, which are related to the Boyer-Lindquist coordinates by the following infinitesimal relations:

$$du_\pm = dt \pm \frac{r^2 + a^2}{\Delta} dr$$

and

$$d\phi_\pm = d\phi \pm \frac{a}{\Delta} dr$$

We note that $u_\pm$ are null coordinates (that is, $\partial_{u_\pm}$ is everywhere null), and the difference of sign indicates a difference of direction: u_+ are incoming in the sense that following a curve of increasing u_+, one crosses the event horizon from the outside (region O) to the inside (region M), allowing us to create matching conditions across this horizon. Similarly, u_- represents outgoing null curves crossing the Cauchy horizon. This interpretation of the $u_\pm$ is borne out by noting that along curves of constant $u_\pm$, we have a simple relation between dt and dr given by the infinitesimal relation above. In particular, we will have the equality of the dr term in this relation with $-dt$, so that:

$$dt = \mp \frac{r^2 + a^2}{\Delta} dr$$

and since as one approaches the Cauchy horizon, integrating, we have t approaching $\mp\infty$, and at the event horizon, approaching $\pm infty$. So the event horizon lies at a point at infinity in the BL coordinates, namely $(r, u_\pm) = (r_1, \mp\infty)$, and the Cauchy horizon at $(r, u_\pm) = (r_0, \pm\infty)$, since we can use the infinitesimal relations above to find that the $u_\pm$ follow t in this way. But then, the Kerr coordinates can be seen to extend the B-L coordinates in the sense that they provide a coordinate system in which the problematic behaviour of the metric at the horizons is eliminated, since the horizons now exist only at points at timelike infinity, hence there are no difficulties crossing the horizon (this terminology is somewhat confusing since it would appear that, as the BL coordinates cross the horizons and the Kerr coordinates do not, the former should be extend the latter - the fact of the coordinate singularity at this crossing accounts for the terminology used).

Now we need to express the Dirac equation in the Kerr coordinates in order to derive the matching conditions we need across the horizons. We remark that, to preserve the form of the Dirac equation, it will be necessary to combine two transformations - both the coordinate change of the Dirac matrices, and a gauge transformation of both the Dirac matrices and the spinors. The general form of the gauge transformation will be:

$$\tilde{\Psi} = V(r)\Psi \tag{5.16}$$

and thus, to preserve the form of the Dirac equation,

$$\tilde{G}^j = V(r)G^j V^{-1}(r) \tag{5.17}$$

Before returning to this, we see how the Dirac matrices transform into the new coordinates, dealing first with the case of the matching across the Cauchy horizon, which is dealt with by the chart (u_+, r, θ, ϕ_+), so we first have:

$$G^{u_+} = G^t \frac{\partial_{u_+}}{\partial t} + G^r \frac{\partial_{u_+}}{\partial r}$$

and

$$G^{\phi_+} = g^{\phi} \frac{\partial \phi_+}{\partial \phi} + G^r \frac{\partial \phi_+}{\partial r}$$

since u_+ depends only on t and r and ϕ_+ depends only on ϕ and r. The other two Dirac matrices remain the same - since we found them in the symmetric frame, however, and this involves a step function of Δ, we find that we have Heaviside functions $\Theta(x)$ (which is 1 for $x \geq 0$ and 0 for $x < 0$) involved in the relevant case, namely that associated with the r-coordinate, which is the coordinate in terms of which Δ is defined.

$$\begin{aligned}
G^{u_+} &= -\frac{a\sin\theta}{\sqrt{U}}\gamma^2 + \frac{r^2+a^2}{\sqrt{U|\Delta|}}(\gamma^0 - \gamma^3) \\
G^r &= -\sqrt{\frac{|\Delta|}{U}}(\Theta(\Delta)\gamma^3 + \Theta(-\Delta)\gamma^0) \\
G^\theta &= -\frac{1}{\sqrt{U}}\gamma^1 \\
G^{\phi_+} &= -\frac{1}{\sin\theta\sqrt{U}}\gamma^2 + \frac{a}{\sqrt{U|\Delta|}}(\gamma^0 - \gamma^3)
\end{aligned} \tag{5.18}$$

With this, we can return to find the gauge transformation matrix $V(r)$ referred to above. This is chosen so as to remove the singularity at the horizon caused by the presence of the Δ in the expressions for G^{u_+} and G^{ϕ_+}. In particular, it may be found that with the transformation given by the matrix

$$V(r) = \frac{1}{2}(|\Delta|^{-\frac{1}{4}} + |\Delta|^{\frac{1}{4}})\mathbb{I} - \frac{1}{2}(|\Delta|^{-\frac{1}{4}} - |\Delta|^{\frac{1}{4}})\gamma^0\gamma^3 \tag{5.19}$$

the problematic forms for G^r and the singularities in G^{u_+} and G^{ϕ_+} disappear, since calculating the new form of the Dirac matrices $\tilde{G}^j$ yields:

$$\begin{aligned}
\tilde{G}^{u_+} &= -\frac{a\sin\theta}{\sqrt{U}}\gamma^2 + \frac{r^2+a^2}{\sqrt{U}}(\gamma^0 - \gamma^3) \\
\tilde{G}^{r} &= -\sqrt{\frac{1}{2\sqrt{U}}}\big((1-\Delta)\gamma^0 + (1+\Delta)\gamma^3\big) \\
\tilde{G}^{\theta} &= -\frac{1}{\sqrt{U}}\gamma^1 \\
\tilde{G}^{\phi_+} &= -\frac{1}{\sin\theta\sqrt{U}}\gamma^2 + \frac{\sqrt{a}}{U}(\gamma^0 - \gamma^3)
\end{aligned} \tag{5.20}$$

In this form, we have matrices which are regular across the horizons, being regular everywhere except at coordinate singularities (and so, via the anticommutation relations, is the metric), and since the Dirac operator $\tilde{G}$ on wavefunctions in this transformed form is constructed as ususal, it too is regular across both horizons: it is just $\tilde{G} = VGV^{-1}$. With this recognized, we can examine its behaviour in the vicinity of the horizons and derive conditions relating the wavefunctions inside and outside them.

We have wavefunctions $\tilde{\Psi}_I$, $\tilde{\Psi}_M$, and $\tilde{\Psi}_O$ which are smooth on the interior of the regions on which they are defined. At the horizons, however, we may have non-smooth behaviour, so it is necessary to consider the total wavefunction $\tilde{\Psi}$ as a generalized function - that is, the Dirac equation holds in the distributional sense for the "function" $\tilde{\Psi} = \tilde{\Psi}_I + \tilde{\Psi_M} + \tilde{\Psi_O}$. In order to capture the effects of these possible discontinuities in the derivatives in the

Dirac equation, we use the Heaviside functions again: near each horizon, we consider $\tilde{\Psi}$ to be locally the sum of the two solutions on each side, multiplied by the appropriate Heaviside function to capture the fact that the function in quesition does not extend past the horizon. The generalized derivatives of the wavefunction then includes Dirac delta functions contributed by the derivative at the horizon. Since the Dirac equation is satisfied by the wave functions everywhere else, this contribution must vanish so that the Dirac equation is satisfied on the whole function. This is what will give us our matching conditions, after we integrate against a test function (to give the distributional equation a meaning).

So, in the case of the Cauchy horizon (for which we have developed the Dirac operator above), looking near $r = r_0$, we have that the wavefunction is locally

$$\tilde{\Psi}(u^+, r, \theta, \phi^+) = \tilde{\Psi}_I(u^+, r, \theta, \phi^+)\Theta(r_0 - r) + \tilde{\Psi}_M(u^+, r, \theta, \phi^+)\Theta(r - r_0)$$

and so the only part of the derivative in the Dirac equation which does not trivially vanish because of our assumption that $\tilde{\Psi}_I$ and $\tilde{\Psi}_M$ are solutions will be the derivative of the Heaviside function - which contribution must also be zero. Since distributional derivatives work in the usual way, this is simply

$$i\tilde{G}^r(\delta_{r_0}(r)\tilde{\Psi}_M - \delta_{r_0}\tilde{\Psi}_I) = 0$$

or, using the form for $\tilde{G}^r$,

$$0 = -\frac{i}{2\sqrt{U}}\delta_{r_0}(r)(\gamma^0 + \gamma^3)(\tilde{\Psi}_M(u^+, r, \theta, \phi^+) - \tilde{\Psi}_I(u^+, r, \theta, \phi^+))$$

When we integrate this (removing the superfluous constant) against some smooth test function $\eta(r)$ in any ε-neighborhood of r_0 to obtain well-defined spinor equations, it becomes

$$0 = \int_{r=r_0-\varepsilon}^{r=r_0+\varepsilon} \eta(r)\delta_{r_0}(r)(\gamma^0+\gamma^3)(\tilde{\Psi}_M(u^+,r,\theta,\phi^+) - \tilde{\Psi}_I(u^+,r,\theta,\phi^+))dr$$

which holds for any ε.

While we may have singular behaviour for the difference $(\gamma^0+\gamma^3)(\tilde{\Psi}_M - \tilde{\Psi}_I)$ as we approach the horizon, this integral will nevertheless be well defined for appropriate choice of test function $\eta(r)$, and since it holds for any test function at all, we may choose such a one to work with. One suitable η, for example, is

$$\eta(r) = \frac{h}{1+|(\gamma^0+\gamma^3)(\tilde{\Psi}_M - \tilde{\Psi}_I)|}$$

since this controls any singular behaviour of the difference term above through the denominator (here, h is any smooth function we may choose).

We observe that, except at the horizon itself, the difference $\tilde{\Psi}_M - \tilde{\Psi}_I$ has one of its terms vanishing. Because of the nonzero contribution to the integral at the horizon due to the delta function δ_{r_0}, and the fact that $eta(r)$ must be smooth here, we cannot conclude from this that for the integral across both sides to cancel both solutions must vanish (though that is in fact the result we shall eventually wish to prove). Instead, we can only obtain a bound on the difference between $\tilde{\Psi}_M$ and $\tilde{\Psi}_I$ as one approaches the horizon. In particular, we obtain an expansion which expresses the difference as some constant plus a correction term. The constant will in general be nonzero, while the first order correction term will be a multiple of $\tilde{\Psi}_M$ - this can be determined from the integral form of the distributional equation above. Thus the matching condition at the Cauchy horizon is:

$$\begin{aligned}
&(\gamma^0+\gamma^3)(\tilde{\Psi}_M(u^+,r+\varepsilon,\theta,\phi^+)-\tilde{\Psi}_I(u^+,r-\varepsilon,\theta,\phi^+))\\
&=(O)(1+|(\gamma^0+\gamma^3)\tilde{\Psi}_M(u^+,r+\varepsilon,\theta,\phi^+))\text{ as }\varepsilon\mapsto 0^+
\end{aligned}\tag{5.21}$$

and the one at the event horizon, similarly, is:

$$\begin{aligned}
&(\gamma^0+\gamma^3)(\tilde{\Psi}_O(u^+,r+\varepsilon,\theta,\phi^+)-\tilde{\Psi}_M(u^+,r-\varepsilon,\theta,\phi^+))\\
&=(O)(1+|(\gamma^0+\gamma^3)\tilde{\Psi}_M(u^+,r-\varepsilon,\theta,\phi^+))\text{ as }\varepsilon\mapsto 0^+
\end{aligned}\tag{5.22}$$

To understand the significance of this condition, we observe that

All of these derivations are exactly parallel the situation in the other Kerr coordinate chart (u_-,r,θ,ϕ_-). The coordinate transformation from BL coordinates to this Kerr coordinate chart give the Dirac matrices to be:

$$\begin{aligned}
G^{u_-}&=-\frac{a\sin\theta}{\sqrt{U}}\gamma^2+\frac{r^2+a^2}{\sqrt{U|\Delta|}}\epsilon(\Delta)(\gamma^0-\gamma^3)\\
G^r&=-\sqrt{\frac{|\Delta|}{U}}(\Theta(\Delta)\gamma^3+\Theta(-\Delta)\gamma^0)\\
G^\theta&=-\frac{1}{\sqrt{U}}\gamma^1\\
G^{\phi_-}&=-\frac{1}{\sin\theta\sqrt{U}}\gamma^2+\frac{a}{\sqrt{U|\Delta|}}\epsilon(\Delta)(\gamma^0-\gamma^3)
\end{aligned}\tag{5.23}$$

which we regularize with the spinorial gauge transformation given by

$$V(r)=\frac{1}{2}(|\Delta|^{-\frac{1}{4}}+\epsilon(\Delta)|\Delta|^{\frac{1}{4}})\mathbb{I}+\frac{1}{2}(|\Delta|^{-frac14}-\epsilon(\Delta)|\Delta|^{\frac{1}{4}})\gamma^0\gamma^3$$

giving the $\tilde{G}^j$ as:

$$\begin{aligned}
\tilde{G}^{u_-}&=-\frac{a\sin\theta}{\sqrt{U}}\gamma^2+\frac{r^2+a^2}{\sqrt{U}}(\gamma^0+\gamma^3)\\
\tilde{G}^r&=-\sqrt{\frac{1}{2\sqrt{U}}}\big((1-\Delta)\gamma^0+(1+\Delta)\gamma^3\big)\\
\tilde{G}^\theta&=-\frac{1}{\sqrt{U}}\gamma^1\\
\tilde{G}^{\phi_-}&=-\frac{1}{\sin\theta\sqrt{U}}\gamma^2+\frac{a}{\sqrt{U}}(\gamma^0+\gamma^3)
\end{aligned}\tag{5.24}$$

(which we note are identical to the $\tilde{G}^j$ for the Kerr +-chart except for the final term of $\tilde{G}^{\phi-}$ and the replacement of $(\gamma^0+\gamma^3$ by $\gamma^0-\gamma^3$ in the first and last cases).

These give matching conditions which are similar to the previous ones in form, except for the change of sign in the γ^3 term. The matching condition at the Cauchy horizon is:

$$\begin{aligned}(\gamma^0-\gamma^3)(\tilde{\Psi}_M(u^-,r+\varepsilon,\theta,\phi^-)-\tilde{\Psi}_I(u^-,r-\varepsilon,\theta,\phi^-))\\ =\mathcal{O}(1+|(\gamma^0-\gamma^3)\tilde{\Psi}_M(u^-,r+\varepsilon,\theta,\phi^-))\text{ as }\varepsilon\mapsto 0^+\end{aligned}\tag{5.25}$$

and at the event horizon:

$$\begin{aligned}(\gamma^0-\gamma^3)(\tilde{\Psi}_O(u^-,r+\varepsilon,\theta,\phi^-)-\tilde{\Psi}_M(u^-,r-\varepsilon,\theta,\phi^-))\\ =\mathcal{O}(1+|(\gamma^0-\gamma^3)\tilde{\Psi}_M(u^-,r-\varepsilon,\theta,\phi^-))\text{ as }\varepsilon\mapsto 0^+\end{aligned}\tag{5.26}$$

In order to make use of these in our subsequent proof of the main nonexistence theorem, however, we will need to have these matching conditions expressed in Boyer-Lindquist coordinates. Since (5.12) shows that the dependence on t and ϕ in the form of the wavefunction is that of a plane wave, being of the form

$$e^{i\omega t-ik\phi}$$

we know that our condition 5.21 must hold in BL coordinates as well, and similarly for the other matching conditions (5.22), (5.25) and (5.26). We will work out the details of the transformation into BL coordinates for the first, noting that they are similar for the others.

Since this condition was obtained by extending across the Cauchy horizon $r=r_0$ at timelike negative infinity at $t=-\infty$, the condition will have the form

$$\begin{aligned}(\gamma^0+\gamma^3)(\tilde{\Psi}_M(t,r_0+\varepsilon,\theta,\phi)-\tilde{\Psi}_I(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|(\gamma^0+\gamma^3)\tilde{\Psi}_M(t,r+\varepsilon,\theta,\phi)|)\text{ across }t=-\infty\end{aligned}\tag{5.27}$$

But now we must convert this condition on the transformed wavefunction in Boyer-Lindquist coordinates into a statement about the untransformed wavefunctions, undoing the gauge transformation to which we subjected the spinors in converting coordinate systems. Recall that we used the transformation 5.16 with the matrix V given by 5.19. But notice that

$$(\gamma^0+\gamma^3)V^{-1}\cong\frac{1}{2}(\gamma^0+\gamma^3)((|\Delta|^{-\frac{1}{4}}+|\Delta|^{\frac{1}{4}})+(|\Delta|^{-\frac{1}{4}}-|\Delta|^{\frac{1}{4}})\gamma^0\gamma^3)\tag{5.28}$$

$$=|\Delta|^{-\frac{1}{4}}(\gamma^0+\gamma^3)\tag{5.29}$$

since the second term vanishes due to antisymmetry of the γ. This means that when we apply the inverse gauge transformation $\Psi=V^{-1}\tilde{\Psi}$ we then can remove this factor of $|\Delta|^{-\frac{1}{4}}$ and get that

$$\begin{aligned}|\Delta|^{-\frac{1}{4}}(\gamma^0+\gamma^3)(\Psi_M(t,r_0+\varepsilon,\theta,\phi)-\Psi_I(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|\Delta|^{-\frac{1}{4}}|(\gamma^0+\gamma^3)\Psi_M(t,r_0+\varepsilon,\theta,\phi)|)\end{aligned}\tag{5.30}$$

But this is not sufficient, since in order to obtain the condition which applies to the form for which the Dirac equation is separable, we must find how this appears for the transformed $\hat{\Psi}=S\Psi$ for the matrix S given by (5.9). This transformation does not affect the factor of the matrix $(\gamma^0+\gamma^3)$, since the only difference in permuting the order in multiplication of S by $(\gamma^0+\gamma^3)$ is the permutation of the blocks of S:

$$\begin{aligned}(\gamma^0+\gamma^3)S=diag((r+ia\cos\theta)^{\frac{1}{2}},(r+ia\cos\theta)^{\frac{1}{2}},\\ (r-ia\cos\theta)^{\frac{1}{2}},(r-ia\cos\theta)^{\frac{1}{2}})\times|\Delta|^{\frac{1}{4}}(\gamma^0+\gamma^3)\end{aligned}\tag{5.31}$$

where the diagonal matrix is just $|\Delta|^{-\frac{1}{4}}S$ with the 2×2 diagonal blocks permuted. We note that this matrix is regular on the horizon, hence has no effect on the matching conditions, so that the only significant effect is the presence of the factor $|\Delta|^{\frac{1}{2}}$, which cancels the reciprocal factor in the form of the matching condition, leaving, in the separable gauge:

$$\begin{aligned}(\gamma^0+\gamma^3)(\hat{\Psi}_M(t,r_0+\varepsilon,\theta,\phi)-\hat{\Psi}_I(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|(\gamma^0+\gamma^3)\hat{\Psi}_M(t,r_0+\varepsilon,\theta,\phi)|)\text{ across }t=-\infty\end{aligned}\tag{5.32}$$

and, through the same transformations (using $(\gamma^0-\gamma^3)$ in the case of the second set of Kerr coordinates), we arrive, for the other matching conditions, at:

$$\begin{aligned}(\gamma^0+\gamma^3)(\hat{\Psi}_O(t,r_0+\varepsilon,\theta,\phi)-\hat{\Psi}_M(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|(\gamma^0+\gamma^3)\hat{\Psi}_M(t,r_0-\varepsilon,\theta,\phi)|)\text{ across }t=\infty\end{aligned}\tag{5.33}$$

$$\begin{aligned}(\gamma^0+\gamma^3)(\hat{\Psi}_M(t,r_0+\varepsilon,\theta,\phi)-\hat{\Psi}_I(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|(\gamma^0-\gamma^3)\hat{\Psi}_M(t,r_0+\varepsilon,\theta,\phi)|)\text{ across }t=\infty\end{aligned}\tag{5.34}$$

$$\begin{aligned}(\gamma^0+\gamma^3)(\hat{\Psi}_O(t,r_0+\varepsilon,\theta,\phi)-\hat{\Psi}_M(t,r_0-\varepsilon,\theta,\phi))\\ =\mathcal{O}(1+|(\gamma^0-\gamma^3)\hat{\Psi}_M(t,r_0-\varepsilon,\theta,\phi)|)\text{ across }t=-\infty\end{aligned}\tag{5.35}$$

These are the matching conditions we shall use in the development of the main theorem which we develop in the next section.

5.2.2 Main Nonexistence Theorem

Our intent in this section is to develop the main theorem proving the nonexistence of (nontrivial) time-periodic solutions of the Dirac equation on the Kerr-Newman background. To do this, we will first prove a technical lemma concerning the 2-spinor X, showing that it has finite values on the event horizon and can be zero there only if it vanishes everywhere outside the horizon (that is, in the region of type O bordering on that horizon).

Since in a physically realistic situation (see [MTW] p882) the surface of a collapsing body would obstruct the boundaries between these regions, and thus these would not be formed by a realistic physical process within the universe, only a part of the full maximal analytic extension is physically relevant. We restrict our attention to some finite subset of the maximal extension, which we call the *physical spacetime*. Any region O (corresponding to the exterior of the black hole) which lies within the physical spacetime is called an *asymptotic end*, and we assume that each such asymptotic end is time-oriented. Each asymptotic end adjoins two regions of type M, one in the past and one in the future. This corresponds intuitively to the notion that one can fall into a black hole, arriving in the future, or that the time-reversed version of this may also occur (thus requiring the matching conditions across the horizon) but that these regions are not the same since, relative to an outside observer, falling into the hole requires an infinite duration, so the past and future regions of type M are separated by an infinite length of time, and are considered distinct.

Since we wish to describe the behaviour of the Dirac wave functions on the physical spacetime, we consider that the wavefunction Ψ vanishes everywhere in the maximal extension which is not included in the physical spacetime, as the rest of the extension is nonphysical, hence the Dirac particle cannot exist there. Similarly, since in the case of a black hole, we assume that Ψ vanishes on regions of type M in the past of the asymptotic ends we are considering (since, matching these solutions across the horizon, we would otherwise have particles emerging from the event horizon, which we are assuming to be impossible).

Since we will be speaking of time periodic solutions, in which the same physical state recurs with some period, we must define what we mean by the same physical state. This will not be defined as the value of the wavefunction, since a physical state must be determined by observables, which correspond to Hermitian operators on the state space $\mathcal{H}$. Because of this, wave functions differing by a constant phase represent the same physical state, and hence time-periodicity must mean that there is a period and a phase difference such that the wavefunction at times separated by that period differ only by the given phase difference. That is, Ψ is time-periodic, of period T if there is some constant Ω such that

$$\Psi(t+T, r, \theta, \phi) = e^{-i\Omega T}\Psi(t, r, \theta, \phi)$$

We remark here that we are defining time-periodicity in terms of the time coordinate in Boyer-Linquist coordinates, since it corresponds to proper time for an observer at infinity, who will observe the time-periodicity of the wavefunction. Now, given such a periodic solution, we can write Ψ as a sum of Fourier coefficients (as with any periodic function), summing over all possible values of the eigenvalue λ, of the spin eigenvalue k, and of the period of the Fourier term. Thus, the wave function decomposes as:

$$\Psi(t, r, \theta, \phi) = e^{-i\Omega t}\sum_{n\in\mathbb{Z}}\sum_{k-\frac{1}{2}\in\mathbb{Z}}\sum_{\lambda\in\sigma_k^n(\mathcal{A})} e^{-2\pi i n\frac{t}{T}}e^{-ik\phi}\Phi^{\lambda n k} \tag{5.36}$$

where $\sigma_k^n(\mathcal{A})$ is the spectrum of $\mathcal{A}$ for fixed values of n and k - that is, the set of possible eigenvalues for the operator $\mathcal{A}$ obtained for those n, k. We remark that $\mathcal{A}$ must have a discrete spectrum since its square can be represented as an elliptic operator on the sphere (see appendix of [FKSY1] for more details). We use the same convention for hatted and unhatted Φ as Ψ, so we have used

$$\Phi^{\lambda nk}(r,\theta) = S^{-1}(r,\theta)\hat{\Phi}^{\lambda nk}(r,\theta)$$

We are using $\hat{\Phi}$ to be of the form given in 5.13:

$$\hat{\Phi}^{\lambda nk} = \begin{pmatrix} X_-^{\lambda nk} Y_-^{\lambda nk} \\ X_+^{\lambda nk} Y_+^{\lambda nk} \\ X_+^{\lambda nk} Y_-^{\lambda nk} \\ X_-^{\lambda nk} Y_+^{\lambda nk} \end{pmatrix}$$

where the $X^{\lambda nk}$ and $Y^{\lambda nk}$ are solutions of the separated ODEs (5.14) and (5.15), where the energy parameter ω is just $\Omega + \frac{2\pi}{T}n$.

The normalization condition is much as described in the case of the fully coupled EDM system: we wish to normalize the scalar product so that $(\Psi|\Psi) = 1$, and therefore must make the requirement that the integral form of this product be finite. As in the fully-coupled system, this cannot in general be done across the horizon, in particular since one cannot choose an everywhere-spacelike hypersurface crossing the horizon. Instead, we must take the inner product associated with surfaces strictly outside the horizon. This, however, means that the current-conservation argument which would make the integral independent of the particular hypersurface fails to work, since we do not cover the whole of the spacetime. Thus, we must restrict attention to one asymptotic end of the physical spacetime and only consider the normalization integral away from the horizon. So, to consider the region outside $r = r_2$, we construct hypersurfaces generated from the Boyer-Lindquist coordinates by:

$$\mathcal{H}_{t_2} = \{(t,r,\theta,\phi) | t = t_2, r > r_2\}$$

and take the inner product defined by the usual normalization integral over $\mathcal{H}_{t_2}$, designated $(\Psi|\Psi)_{\mathcal{H}_{t_2}}$. This gives the probability of the Dirac particle being outside $r = r_2$, which must naturally be finite. So the normalization condition would be

$$(\Psi|\Psi)_{\mathcal{H}_{t_2}} = \int_{\mathcal{H}_{t_2}} \overline{\Psi} G^j \Psi \nu_j d\mu < \infty \text{for all} t_2$$

If the Dirac wave function satisfies this condition, then we want to show that each Fourier component of it must also satisfy the same normalization condition, so that we can restrict our analyses to static solutions. Integrating the given condition with respect to time, to average the inner product over one whole period of the whole wave function, we get:

$$\infty > \frac{1}{T} \int_t^{t+T} (\Psi|\Psi)_{\mathcal{H}_\tau} d\tau \tag{5.37}$$

$$= \sum_{n,n'} \sum_{k,k'} \sum_{\lambda,\lambda'} \frac{1}{T} \int_t^{t+T} e^{-2\pi i (n'-n)\frac{\tau}{T}} d\tau \tag{5.38}$$

$$\times \int e^{-i(k'-k)\phi} \overline{\Phi^{\lambda n k}(r,\theta)} \Phi^{\lambda' n' k'}(r,\theta) d\mu_{\mathcal{H}} \tag{5.39}$$

But note that since this is integrated over a whole period, and the plane waves form an orthogonal set, the only nonzero terms come from those cases where $n = n'$ and $k = k'$, while the integration over θ (see the appendix of [FKSY1] for the regularity of the angular part) is nonzero only when $\lambda = \lambda'$, hence we can collapse this form by eliminating the exponentials (the first integral becomes T and cancels the $\frac{1}{T}$) to get:

$$\sum_{n \in \mathbb{Z}} \sum_{k+\frac{1}{2} \in \mathbb{Z}} \sum \lambda \in \sigma_k^n(\mathcal{A}) \int_{\mathcal{H}_{t,r_2}} \overline{\Phi^{\lambda n k}(r,\theta)} \Phi^{\lambda n k}(r,\theta) d\mu_{\mathcal{H}}$$

Now the integral in this last form is just the scalar product $(\Phi^{\lambda n k}|\Phi^{\lambda n k})_{\mathcal{H}_t}$, and since this product is positive, each of these contributing terms is positive and therefore finite since their sum is finite. Since we can therefore restrict our attention to a single Fourier component and obtain results which will hold for any time-periodic solution. We begin by showing a boundedness result for the spinor X which is somewhat analogous to the result on $(\alpha^2+\beta^2)$ in the case of the fully coupled spherically symmetric case.

Lemma 4. *The function* $X = \begin{pmatrix} X_+ \\ X_- \end{pmatrix}$ *has finite squared norm* $|X|^2$ *on the event horizon, and if it is zero at* $r = r_1$ *then* X *vanishes for all* $r > r_1$.

Proof. Recall the Dirac equation's radial component, (5.14) governing the function X, namely

$$\begin{pmatrix} \sqrt{|\Delta|}\mathcal{D}_+ & imr - \lambda \\ -imr - \lambda & \epsilon\Delta\sqrt{|\Delta|}\mathcal{D}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0$$

which yields, for all $r > r_1$, an ODE for $|X|^2$, namely:

$$\begin{aligned} \sqrt{|\Delta|}\frac{d}{dr}|X|^2 &= \sqrt{|\Delta|}\frac{d}{dr} < X, X > \\ &=< \sqrt{|\Delta|}\frac{d}{dr}X, X > + < X, \sqrt{|\Delta|}\frac{d}{dr}X > \\ &= 2\lambda Re(\overline{X_+}X_-) + 2mrIm(\overline{X_+}X_-) \end{aligned} \tag{5.40}$$

where the last equality follows from the eigenvalue equation found for X. But this gives bounds on the radial derivative of $|X|^2$, since we have it equal to the sum of the real and imaginary parts of the same function, multiplied by constants. This gives the bound

$$\left|\sqrt{|\Delta|}\frac{d}{dr}|X|^2\right| \le (|\lambda| + mr)|X|^2$$

So we note that if $|X|^2$ vanishes anywhere for $r > r_1$, then since its derivative vanishes there as well, by this bound, and since X is a solution to this ODE, then the solution (which is unique by standard theory of ODEs) is that

$X \cong 0$. We must now consider the limiting behaviour as $r \mapsto r_1$. Suppose that $|X|^2 > 0$ outside $r = r_1$. Then we can divide the bound above by $\sqrt{|\Delta|}|X|^2$, to obtain

$$-(|\lambda| + mr)|\Delta|^{-\frac{1}{2}} \leq \frac{\frac{d}{dr}|X|^2}{|X|^2} \leq (|\lambda| + mr)|\Delta|^{-\frac{1}{2}}$$

noting that the middle expression is the derivative of $\log |X|^2$, we integrate this bound to find:

$$-\int_r^{r'} (|\lambda| + mr)|\Delta|^{-\frac{1}{2}} \leq \log |X|^2 \Big|_r^{r'} \leq \int_r^{r'} (|\lambda| + mr)|\Delta|^{-\frac{1}{2}}$$

But while $|\Delta|^{-\frac{1}{2}}$ is singular at r_1, it is integrable (since Δ is a quadratic in r), so this implies that $\log |X|^2$ has a finite limit at r_1 in any case (since if it is zero anywhere, it will have the finite limit of 0 at r_1 as well). This proves the lemma. □

With this lemma, we can proceed to the main theorem under consideration.

Theorem 2. *In the background of a Kerr-Newman black hole which is non-extreme (i.e. for which $a^2 + Q^2 < M^2$), there are no nontrivial normalizable time-periodic solutions for the Dirac Equation.*

Proof. We proceed by showing that any such solution must vanish everywhere in every region of the physical spacetime. The causal structure of the maximal extension gives a natural sequence in which to do this. We begin with regions of type O, since if we assume that nothing can leave the event horizon, this is independent of the form of any solution elsewhere. We thus begin by showing that in each asymptotic end of spacetime, $\Psi \cong 0$.

Recall that we have made the assumption that, since we are dealing with a black hole solution, we assume that $\hat{\Psi}_M \cong 0$. We can apply the matching condition (5.35) across the horizon, so that:

$$(\gamma^0 - \gamma^3)(\hat{\Psi}_O(t, r_1 + \epsilon, \theta, \phi)) = \mathcal{O}(1)$$

across $t = -\infty$, which in terms of the radial functions means

$$\lim_{r \mapsto r_1^+} X_-(r) = 0$$

Now using the relation between the hatted and unhatted form of the wave-function,

$$\overline{\Psi}\gamma^0\Psi = |\Delta|^{-\frac{1}{2}} U^{-\frac{1}{2}} \overline{\hat{\Psi}}\gamma^0\hat{\Psi}$$

and the fact that the metric is asymptotically flat, we find that the normalization condition simply becomes a condition on the integral of the function $|X|^2$, namely

$$\int_{r_2}^{\infty} |X|^2 dr < \infty$$

for any $r_2 > r_1$. But outside the horizon (where $\epsilon(\Delta) = 1$) the radial Dirac equation 5.14, when expanded out, gives two opposing terms summing to zero, which are just

$$\frac{d}{dr}(|X_+|^2 - |X_-|^2) = 0$$

(This last statement is equivalent to the statement that the Dirac current $\overline{\Psi}G^r\Psi$ in the radial direction is a conserved quantity, since $\overline{\Psi}G^r\Psi = U^{-\frac{1}{2}}(|X_+|^2 - |X_-|^2)|Y|^2)$. But this means that this function $|X_+|^2 - |X_-|^2$is constant. But the normalization condition above means that the integral of $|X_+|^2 + |X_-|^2$ is finite, so if the (constant) difference were nonzero, the sum would be at least as large, hence its integral over a spacelike hypersurface (having infinite volume) would be infinite. So in fact $|X_+|^2 - |X_-|^2 \cong 0$. But since $\lim_{r \mapsto r_1^+} X_-(r) = 0$, then the same must hold for X_+, so that X is zero on the horizon, and by lemma 4 vanishes everywhere. By the ansatz for $\hat{\Psi}$, this means that $\hat{\Psi}_O \cong 0$ also (that is, any time periodic solutions of the Dirac equation vanish outside the event horizon).

Now we consider a region of type M, between the Cauchy and event horizons. Regions of type M border on regions of type O across an event horizon in both the past and future directions, and since in these regions the wavefunction vanishes, we get the matching conditions across these horizons to be:

$$\lim_{r \mapsto r_1^+} (\gamma^0 + \gamma^3)\hat{\Psi}_M(t, r, \theta, \phi) = 0 = \lim_{r \mapsto r_1^+} (\gamma^0 - \gamma^3)\hat{\Phi}_M(t, r, \theta, \phi)$$

But then these together imply that the wavefunction itself must vanish in this limit, since it does so when multiplied by either γ^0 or γ^3, so that:

$$\lim_{r \mapsto r_1^+} \hat{\Phi}_M(t, r, \theta, \phi) = 0$$

But then by 5.14, we will have the radial derivative of the squared norm of the X spinor vanishing, or in particular

$$\sqrt{|\Delta|}\frac{d}{dr}|X|^2 = 0$$

in the regions of type M, but as seen in lemma (4), this means that $|X|^2$ must vanish everywhere if it is to be zero at the horizon. Thus, Ψ must vanish in regions of type M.

This leaves only regions of type I to consider. These meet regions of type M at the Cauchy horizon $r = r_0$ at both past and future infinity - in Boyer-Lindquist coordinates, at $t = \pm\infty$. We have already seen that in regions of type M, the wavefunction Ψ vanishes, so that the matching conditions across the Cauchy horizon (5.32) and (5.34) again imply the limit of the wavefunction near that horizon vanishes:

$$\lim_{r \mapsto r_0^+} \hat{\Psi}_I(t, r, \theta, \phi) = 0$$

But now regions of type I and O are symmetric, and the radial Dirac equation (5.14) is the same in each region, and so lemma (4) applies again, hence X must vanish everywhere, and thus $\hat{\Psi}_I \cong 0$. Thus, since $\hat{\Psi}$ vanishes in each of the three types of region, we have shown the result of the theorem. $\square$

This theorem has shown that any one of the Fourier components of a time periodic solution to the Dirac equation on the Kerr-Newman background must vanish everywhere, which thus implies that the solution itself must do so as well. This is the last result we wish to show. We remark only that it can be generalized to more general metrics in which the Dirac equation is separable in the same way as in the KN geometry, which occurs in metrics in which the Weyl conformal curvature tensor has *type D*, meaning that it have two repeated eigenbivectors. For more detail on this, refer to part 3 of [FKSY1].

Appendix A

Background on the Dirac Equation

The Dirac equations, which are the equations governing such behaviour, make use of spinor fields to describe a certain class of fields having nontrivial spin characteristics. Although it is possible to describe these fields in purely tensorial terms, the calculations are greatly simplified by using the spinorial formulation. These fields correspond, for instance, to electrons or neutrinos in physical situations. To make clearer the description of the Dirac equation, we begin with a consideration of spinors in curved spacetime.

The usual definition of spinors used by physicists describes them by saying that a spinor at a point x on a manifold M is an equivalence class of pairs (ψ, ρ), where ψ is a complex 2-vector and ρ is an orthonormal basis of the tangent space M_x. The equivalence is given by:

$$(\psi, \rho) \sim (\psi', \rho')$$

if

$$\rho' = L\rho, \psi' = \Lambda\psi, L = \mathcal{H}\Lambda$$

where L is a Lorentz transformation and Λ is an element of $Spin(4)$, which we shall describe next, and $\mathcal{H}$ is the homomorphism we shall give from $Spin(4)$ onto the group $L(4)$ of Lorentz transformations of T_xM.

Given an inner-product space $(V, <,>)$, it is possible ([Har]) to define the *Clifford Algebra* $Cl(V)$ as $\bigotimes V/I(V)$, where $I(V)$ is the ideal in $\bigotimes V$ generated by elements of the form $x \otimes x + < x, x >$. This gives (as in [Cho]) an algebra of linear operators on a complex vector space, generated by elements satisfying

$$\gamma_u \gamma_v + \gamma_v \gamma_u =< u, v > e$$

where e is the identity operator in $Cl(V)$. If we take the inner product space in question to be the Minkowski inner product, these γ are linear operators on a complex vector space having the property that $\frac{1}{2}(\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \eta_{\alpha\beta} \cdot e$. On curved spacetime, we will take the more general metric $g_{\alpha\beta}$ instead of $\eta_{\alpha\beta}$. Given a basis e_0, e_1, e_2, e_3 of the Minkowski space $T_x(M)$, such an algebra is generated (as an algebra) by the basis of Dirac matrices:

$$\gamma_{e_0} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \gamma_{e_i} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Here the σ_i are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

We can then define the group $Spin(4)$, which will be the transformation group for Dirac spinors:

Definition 1. The group $Spin(4)$ is the group of real linear operators Λ, of unit determinant, on a complex vector-space of dimension 4, such that if

$u \in T_x(M)$, then there exists $v \in T_x(M)$ such that $\Lambda^{-1}\gamma_u\Lambda = \gamma_v$. That is, the Λ are unit determinant operators fixing the Clifford algebra $Cl(T_x(M))$.

It may be shown that $Spin(4)$ is the universal covering group of the Lorentz group $L(4)$, and that the quotient $Spin(4)/L(4) \cong \mathbb{Z}/2\mathbb{Z}$. In particular, the covering gives a homomorphism of $Spin(4)$ onto $L(4)$, which we have denoted $\mathcal{H}$ previously. For any element $L \in L(4)$, there are two elements, Λ and $-\Lambda$, whose image under $\mathcal{H}$ is L. This is related to the fact that $L(4)$ has the nontrivial homotopy group $\mathbb{Z}/2\mathbb{Z}$ (the quotient mentioned above): the unique nontrivial homotopy class is that of a path taking a basis through a rotation of 2π about the origin and returning to the initial position.

We are now in a position to discuss the spin connection and the Dirac operator. The Dirac operator will be a partial differential operator of the form $(G - m)$ acting on spinor fields: solutions to the Dirac equation will be fields Ψ for which $(G - m)\Psi = 0$. Here, m is the mass of the field (which may be zero); to define the G part of the operator, it is necessary to define a spin connection - in other words, to do geometry on the spin bundle of spacetime. Differential geometry is essentially concerned (cf. the treatment in [Sha]) with the study of connections on a principal bundle: relevantly here, a connection provides a notion of covariant differentiation on a manifold. In general, a connection on a principal bundle is a 1-form with values in the Lie algebra of the structure group. One standard example is the case of a locally Minkowskian manifold, and the principal bundle of frames, with structure group $L(4)$: with each direction in the tangent space to the manifold, a connection associates an element in the tangent space at the identity of the Lie group of basis transformations. In other words, the connection describes how a basis is parallel-transported along a curve: this leads immediately to the parallel transport of vectors and covariant differentiation. The spin derivative is related to this notion, but the principal bundle is the $Spin(4)$ bundle on M.

(We remark here that the notion of a $Spin(4)$ bundle on M need not be well defined for arbitrary manifolds M. There is a topological obstruction to the construction of such a bundle which relates to the Stiefel-Whitney class of the manifold, which plays an analogous role for real bundles to the role of the Chern class in complex bundles. It is a characteristic class in the cohomology group of $T(M)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Though a detailed discussion of this is not relevant here, we remark upon it to make clear that the requirement that a spin-bundle should be defined on M gives a topological condition on M. This condition is that the second Stiefel-Whitney class $w_2M(T(M))$ should be zero.)

We note that since there is a canonical homeomorphism from $Spin(4)$ to $L(4)$, a connection ∇ on the $L(4)$ bundle can be pulled back to a connection $\mathbf{D}$ on the $Spin(4)$ bundle: the pulled-back 1-form acts on an element Λ of $Spin(4)$ by letting $\mathbf{D}(\Lambda) = \nabla(L)$, where $L = \mathcal{H}(\Lambda)$. This gives a well-defined spin connection, defining a spin derivative, which can be used to construct differential operators acting on spinor fields, and in particular the Dirac operator which is of concern here. We present a brief summary of this development; for a fuller description, see e.g. [Fin].

The G in the Dirac Operator noted above is a partial differential operator given in terms of the spin derivative by $G = iG^j\mathbf{D}_j + H$ where $H = H(x)$ is a self-adjoint matrix at each point $x \in M$. Here, the G_j are Dirac matrices described above, which locally look like the standard Dirac matrices γ_j, defined in curved spacetime by the property that $\frac{1}{2}\{G_i G_j\} = g_{ij} \cdot e$ (where $\{\}$ denotes anti-commutation). This expression is fairly general: to do explicit calculations, however, it is necessary to write the spin derivative in terms of known entities. This is analogous to expressing the covariant derivative in terms of directional derivatives and Christoffel symbols. Thus, for the purposes of calculation, we write the term G in the Dirac operator in the form (see [Fin] or [FSY1]):

$$G = iG^j \frac{\partial}{\partial x^j} + B(x)$$

where the $B(x)$ are 4×4 matrices playing a role analogous to that of the Christoffel symbols in covariant differentiation. In [Fin], it is shown that they have the form:

$$B(x) = G^j(x) E_j(x) \tag{A.2}$$

with

$$E_j = \frac{i}{2}\rho(\partial_j \rho) - \frac{i}{16} Tr(G^m \nabla_j G^n) G_m G_n + \frac{i}{8} Tr(\rho G_j \nabla_m G^m)\rho \tag{A.3}$$

and the symbol ρ, which in flat spacetime is sometimes denoted γ^5 by analogy with the other γ matrices, has the form (with ϵ_{ijkl} the volume form)

$$\rho = \frac{i}{4!}\epsilon_{ijkl} G^i G^j G^k G^l \tag{A.4}$$

These expressions are clearly rather complicated, and difficult to make use of unless special symmetry properties of the metric allow simplification. The examples studied in Part II are instances of cases in which this occurs.

Appendix B

Topological Methods for DEs

The use of topological methods in studying differential equations dates back to the invention of topology itself, by Poincare. One branch of such methods includes *degree theory*, which seeks to understand the structure of solutions to such equations by the use of a “degree” of a function, which measures the stability of that function, its critical points, and related features, in an open neighborhood. In order to make clear the application of this theory to the system we consider in this work, we shall briefly describe some of the techniques and principles of these methods.

We begin with Morse theory, which is used to study the topological properties of, and in particular the stability of critical points of, gradient fields, and the flows corresponding to them. The “Morse Index” developed there is a measure of the attractiveness of a critical point to flows, and hence its “stability”; this index is invariant under small changes in the gradient. The Morse Index can be generalized to more general fixed sets, by passing from a numerical index to a topological invariant, in what is known as the “Conley Index”, which is the tool which we apply to our solutions of the

Einstein-Dirac equation in Part I. These techniques, being qualitative and having discrete values for their "indices", have the attractive property of not requiring exact calculations, and thus being readily applicable to our rough numerical solutions, subject to certain conditions.

A fuller development of these methods can be found in [Smo], but here we begin our presentation with a brief examination of Morse Theory.

We are here considering gradient fields, which are fields of the form df for f at least in $C^2(\Omega, \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$ is an open neighborhood. For such a field, $\overline{x}$ is a critical point if $df(\overline{x}) = 0$. We note that the property of being a critical point is preserved under smooth maps $x = \phi(y)$ of $\mathbb{R}^n$, so that if $F = \phi \circ f$ and $\overline{x} = \phi(\overline{y})$ then $\overline{y}$ is a critical point of dF. Note that this means that the theory can in fact be developed on any smooth (C^∞) manifold, which is essential for our desired application. The Morse index allows us to examine the structure of such points, which we may think of as fixed points of the flow determined by df. We must restrict our attention to isolated critical points: that is, those for which there are no other critical points in some neighborhood of the point in question. If all critical points of f are isolated, f is a *Morse function.*

Restricting our attention to Morse functions is not a serious limitation, since they are generic in $C^2(\Omega, \mathbb{R})$ for any neighborhood Ω (that is, they form a dense open set in $C^2(\Omega, \mathbb{R})$). This means that the study of Morse functions is quite general, and the following normal form theorem is widely applicable. This theorem, for functions with non-degenerate critical points, states essentially that near any such point, the function is an n-dimensional "saddle" point with some number of dimensions taken by axes in which f

decreases (in both directions), and some number with directions in which it increases - these numbers giving the index. A more precise form of this is the following:

Theorem 3. *Given a function $f \in C^2(\Omega, \mathbb{R})$ with non-degenerate critical point $\overline{x}$, there is a coordinate system near $\overline{x}$ such that*

$$f(x) = f(\overline{x}) + \sum_{1}^{n} \epsilon_i x_i^2$$

where $\epsilon_i = \pm 1$.

The number k of positive eigenvalues (which is independent of the coordinate system since it is a property of the Hessian of f), will be the Morse index of the critical point $\overline{x}$, and can be thought of in terms of stable and unstable manifolds (which will lead to the generalization in the Conley index). As has already been remarked, the Morse index addresses the stability of flows of the system $\frac{dx}{dt} = \nabla f(x)$, for which the critical point $\overline{x}$ is a stationary point. To make this precise, we recall the following definitions and theorem:

Theorem 4. *(Linear Stable Manifold Theorem) Given the setup just described, there are manifolds M_k (the unstable manifold), and M_{n-k} (the stable manifold) of dimensions k and $n-k$, such that*

- *if $y_0 \in M_k$ them $\phi_t(y_0) \mapsto \overline{x}$ as $t \mapsto -\infty$*
- *if $y_0 \in M_{n-k}$ them $\phi_t(y_0) \mapsto \overline{x}$ as $t \mapsto \infty$*

Here, recall that $\phi_t(y_0)$ denotes the point to which y_0 will flow at time t in the system $\frac{dx}{dt} = \nabla f(x)$. The stable manifold is thus defined to be the set of points on flows which asymptotically approach the critical point in future time, and the unstable manifold, similarly in past time. The theorem asserts that these manifolds exist in sufficiently small neighborhoods of the critical point. We remark that these definitions (and theorem) justify our description of the Morse index k as a measure of stability, since it is the dimension of the unstable manifold - that is, of the surface of points which asymptotically "flee" the critical point. By examining this situation in terms of these manifolds, we are beginning to approach the topological definition of the index which shall be the basis of our generalization to larger sets than points.

The LSM theorem implies that there is some open set, say B, about $\overline{x}$ which intersects M_k and M_{n-k} in, respectively, a k-ball and a $(n-k)$-ball, which we denote B^k and B^{n-k}, which have boundaries in ∂B which are a $(k-1)$-sphere and an $(n-k-1)$-sphere. Note that B can be regarded naturally as $B^k \times B^{n-k}$. If we consider points on ∂B as (possibly) entrance or exit points of flows (assuming flows do not remain in ∂B for nonzero time), then we define:

$$b^- = \{x \in \partial B : \exists \epsilon > 0 : \phi_{(-\epsilon,0)}(x) \cap B = \emptyset\}$$

and

$$b^+ = \{x \in \partial B : \exists \epsilon > 0 : \phi_{(0,\epsilon)}(x) \cap B = \emptyset\}$$

so that we can regard b^+ as $\partial B^k \times B^{n-k}$ and b^- as $B^k \times \partial B^{n-k}$. We will be primarily interested in the space B/b^+, namely the space obtained by collapsing b^+ to a point, and in particular, we will be interested in the homotopy class of this space. The space B/b^+ is called the *topological Morse index* of f at $\overline{x}$. To see how it relates to the Morse index defined previously, suppose that f is a function with nondegenerate critical point $\overline{x}$ having Morse index k: then if we take the coordinate system in which

$$f(x) = f(\overline{x}) + \sum_{1}^{n} \epsilon_i x_i^2$$

and let B be the preimage in this coordinate chart of a sufficiently small cell $(-\delta, \delta) \times \cdots \times (-\delta, \delta)$ centered about the origin (of which $\overline{x}$ is the preimage), then b^+ will be the sides of ∂B corresponding to the positive ϵ_i, since along those directions, f is increasing, and hence the flows exit B. Contracting b^+ to a point, then, we obtain a "fattened" figure with the same homotopy type as a k-sphere S^k. Thus, the topological Morse index for a point with classical Morse index k is the homotopy type of S^k.

Notice that in the preceding construction, only the values of f and df around ∂B are relevant to the determination of the index: this implies both that the index should be invariant under small perturbations and that it should not matter that the fixed set contained in B happened to be the unique fixed point $\overline{x}$. The Conley index is an attempt to convey information about the stability of fixed sets in much the same way that the Morse index does for fixed points. In order to make this precise, we must define the type of sets which we shall consider. First, we will generalize to arbitrary differential equations $\frac{dx}{dt} = f(x)$, rather than restricting ourselves to gradient fields (which was necessary to define a nondegenerate critical point, which we no longer need to do).

Definition 2. A set is an *invariant set* if it is a union of solution curves $\{\phi_t(x) : t \in \mathbb{R}\}$ - hence it is fixed under flows both backwards and forwards in time. An invariant set S is *isolated* if there is some bounded neighborhood N

of S, called an *isolating neighborhood* if compact, such that S is the maximal invariant set in N.

Isolated invariant sets are of interest since, there being no other invariant sets "near" them in precisely the given sense, they are stable under small perturbations in f below. That is, since any nearby flow not in the isolated invariant set must leave N in either past or future time, this must continue to be true for functions nearby to f (in the compact-open topology), since N is an isolating neighborhood precisely if no point on its boundary is on a solution curve contained in N, which is a property preserved under sufficiently small perturbations. This leads to the concept of a *continuation* of S, which we shall define briefly after introducing a few necessary concepts (for a more detailed treatment, see e.g. [Smo] pp460-461).

Definition 3. Given a flow on a space M, $X \subset M$ is a *local flow* if for each point $\gamma \in X$, there is a neighborhood U of γ and some $\varepsilon < 0$ such that the image of U under the homeomorphism ϕ_t is in X for $t \in [0, \varepsilon)$. A *product parameterization* of a local flow X is a homeomorphism ϕ from $X_1 \times \Lambda$ into X such that for every $\lambda \in \Lambda$ we have $X_\lambda \cong \phi(X_1 \times \lambda)$ is a local flow in X.

We can think of a product parameterization of a local flow as a flow which depends upon some parameter - this is the origin of the notion of nearby flows, from which we derive the idea of continuation, which requires the concept of the space of isolated invariant sets of a parametrized local flow of this kind, to wit:

Definition 4. If $\phi : X_1 \times \Lambda \mapsto X$ is a product parametrization of a local flow X, define $\mathcal{S}$ to be $\mathcal{S}(\phi) = \{(S_\lambda, X_\lambda) : S_\lambda \text{is an isolated invariant set of } X_\lambda\}$

This gives the notion of a continuation of an isolated invariant set, namely:

Definition 5. If p_1 and p_2 are points in $\mathcal{S}$ then p_1 is a *continuation* of p_2 (or p_1 and p_2 are *related by continuation*) if both p_1 and p_2 lie in the same quasi-component of $\mathcal{S}$ (that is, it is not the case that $\mathcal{S}$ is the disjoint union of two open sets each containing one point).

This definition allows us to follow isolated invariant sets through different, related flows deriving from a parametrized PDE - in the context of the present work, the parameter is the fermion mass m. This is primarily useful because we can relate the Conley Index of such isolated sets to those of other such sets, which allows us to understand properties of the stability of many solutions at once. To see how this is accomplished, we examine the definition of the Conley index more closely. The Conley Index is developed in the context of *isolating blocks*, a special type of isolating neighborhood characterized by having no points on the boundary which remain there under the action of the flow. In particular, we call a subset S of X a *local section* of a flow if the flow, for short times δ, defines a local homeomorphism $h_\delta : S \times (-\delta, \delta) \mapsto X$. Note that this cannot occur if the orbits of the flow are tangent to S since near such a point, the "fattened" S will be self-intersecting. We can then define an isolating block by

Definition 6. B is called an isolating block for a flow f on X if B is the closure of a neighborhood of X, and $S^{\pm}$ are two open sections such that:

1. $cl(S^{\pm}) \setminus S^{\pm} \cap B = \emptyset$
2. $S^{-} \cdot (-\delta, \delta) \cap B = (S^{-} \cap B) \cdot [0, \delta)$

3. $S^+ \cdot (-\delta, \delta) \cap B = (S^+ \cap B) \cdot (-\delta, 0]$

4. If $x \in \partial B \setminus (S^+ \cup S^-)$ then there are $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ with

We remark that S^- intersects B just at the boundary, at points where the flow enters B, while S^+ is the same, but for exit points. Note also that an isolating block is just a special type of isolating neighborhood. It is a useful fact (since the Conley index will be defined via isolating blocks) that one can always be found about an isolated invariant set, and in fact any neighborhood of such a set contains an isolating block about that set.

We now have the language to understand what is meant by an index: namely, it is a function constant on compontents of $\mathcal{S}$, or in other words invariant under continuation. The Conley Index, in particular, though it is defined in terms of isolating blocks, can be shown to be independent of them, and to be well defined on such components - it will indeed be an index. It is defined by:

Definition 7. If I is an isolated invariant set of a flow, and B an isolating block of I, the Conley Index of I is $h(I) \cong [B/b^+]$, the homotopy equivalence class of B/b^+ considered as a pointed space.

We remark that this is indeed a well defined property of I since if B_1 and B_2 are two isolating blocks for I, then $B_1/b_1^- \sim B_2/b_2^-$ and $B_1/b_1^+ \sim B_2/b_2^+$. (For details of this, see for instance [Smo] pp475-476). A useful fact about the Conley index is given by the following theorem:

Theorem 5. *If I_1 and I_2 are isolated invanriant sets and $I_1 \cap I_2 = \emptyset$ then $I_1 \amalg I_2$ is an isolated invariant set whole Conley Index is*

$$h(I_1 \amalg I_2) = h(I_2) \wedge h(I_2)$$

(where $\wedge$ denotes the wedge product between two pointed spaces, obtained by taking the disjoint union and identifying the distinguished points). This theorem gives an "addition" for the Conley index for which the identity is the homotopy type of the one-point space, $(X, x_0) = (\{x_0\}, x_0)$ which we designate $\overline{0}$.

This makes the Conley index well defined for a particular isolated invariant set, and allows us to find the index of collections of such sets. In fact, we have rather more than this - in particular, we have that h defines an index as defined above. Namely:

Theorem 6. *If I_λ and I_μ are related by continuation, then they have the same Conley index.*

This is the key result which makes the stability arguments in Part I possible, allowing us to extend the index of stable sets representing solutions around the spiral form by continuation and draw conclusions about the stability of these solutions from the shape of the curve and the index of the case with low m.

We remark here that it is possible to define the Conley index in a more general way, in terms of *index pairs*, and that this redefinition, while equivalent with the one we have given here, is used in the development of the last theorem. In the interests of clarity and brevity, however, we have omitted this part of the development of the index theory. This development is given more fully in [Smo], chapters 22 and 23.

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