

EXTENDED TQFT, GAUGE THEORY, AND 2-LINEARIZATION

JEFFREY C. MORTON

ABSTRACT. In this paper, we describe a relation between a categorical quantization construction, called “2-linearization”, and extended topological quantum field theory (ETQFT). This is a weak 2-functor $Z_G : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$ valued in (Kapranov-Voevodsky) 2-vector spaces. The 2-linearization process assigns 2-vector spaces to (finite) groupoids, and structure-preserving functors between them to spans of groupoids. By combining this with a topological gauge theory with finite gauge group G , the ETQFT obtained is the (untwisted) Dijkgraaf-Witten model associated to G , extended to manifolds with boundary as described by Freed. This construction is related to the “quantization” functors of Freed, Hopkins, Lurie and Teleman. We give some explicit examples and calculations of invariants, and described some consequences. We then describe how to extend the 2-linearization process to accommodate the twisted DW model.

1. INTRODUCTION

This paper demonstrates a construction of an extended topological quantum field theory (ETQFT), associated to any finite group G , by means of a 2-functor Λ , called “2-linearization”, described by the author in [36], and a “twisted” form of Λ introduced here. The 2-functor Λ takes finite groupoids to 2-vector spaces, spans of such groupoids to 2-linear maps, and spans of spans to natural transformations. The twisted variant behaves similarly for a 2-category where the groupoids carry some cocycle data. The ETQFT Z_G defined here is related to a topological gauge theory and in particular the Dijkgraaf-Witten model.

This paper has three main objectives: to show how Λ gives a categorical interpretation of well-known constructions; to provide a physically-motivated interpretation of Λ ; and to draw on this motivation to see how to generalize Λ to the more physically meaningful “twisted” case.

For the first objective, the example we obtain in this way derives from the original work of Dijkgraaf and Witten [13], and Freed and Quinn [17], as well as elsewhere. The characterization Λ in terms of ambi-adjoint functors in [36] gives a conceptual account of various normalization factors, and organizes the physical structure into a very general structure. It also shows how two levels of these constructions, at codimensions one and two, are precisely parallel, with one being a “categorification” of the other.

For the second and third objectives, the point here is that 2-linearization is to be understood as a categorification of the path integrals (in the discrete case, “sums over histories”) which are used in the construction of the DW model.

2010 *Mathematics Subject Classification.* 18E10, 20L05, 57R56.

Key words and phrases. TQFT, groupoids, 2-vector spaces, gauge theory.

Recall that Atiyah’s axioms for an n -dimensional TQFT describe it as a symmetric monoidal functor

$$(1) \quad Z : \mathbf{nCob} \rightarrow \mathbf{Hilb}$$

where \mathbf{nCob} is a category whose objects are $(n-1)$ -manifolds and whose morphisms are cobordisms.

In general, a “ k -tuply extended TQFT” assigns higher-categorical structures called k -vector spaces to manifolds of codimension k . In particular, it is a (weak, monoidal) k -functor:

$$(2) \quad Z : \mathbf{nCob}_k \rightarrow \mathbf{k-Vect}$$

(The relevance of 2-vector spaces to the setting of topology as in ETQFT has been described in more detail, for example, by Yetter [46].) In this paper we are only interested in the case $k = 2$, though the construction given here might be generalized to higher k . That is, we are interested in considering the ETQFTs which act on manifolds with boundary, and cobordisms between these.

This two-tier structure illustrates the program of Freed [15] on the use of higher-algebraic structures for quantization, which in turn integrated into a beautiful higher-algebraic framework by Freed, Hopkins, Lurie and Teleman [16]. That work refers to “canonical quantization for classical field theories valued in n -groupoids”. Spans (or “families”) are treated somewhat differently than they are here. In particular, the construction of Λ makes essential use of two-sided adjunctions, which are due to universal properties of the span construction.

The physical meaning of our program, agrees with the framework of [16], in which there are two ingredients to an ETQFT:

- (1) A “classical” field theory, where the values of the fields live in an n -groupoid.
- (2) A “quantization functor” which takes the n -groupoids to $(n+1)$ -algebras and spans to morphisms of all degrees.

The 2-functor Λ which we use here fits the second half of the framework of [16] where the target is an n -category of n -algebras and bimodules. We have not explicitly emphasized the monoidal structure on our 2-vector spaces (the tensor product of representations) which makes them 2-algebras, nor presented the morphisms of n -algebras as bimodules. However, 2-linear maps correspond exactly to bimodules, via a slightly indirect construction: each component of the matrix representation, associated to a pair of irreps, gives a multiplicity, and can be turned into a bimodule by tensoring on each side with the irreducible representations in question. Schur’s lemma then ensures that tensoring with this bimodule amounts to applying the 2-linear map.

The main result of this paper can be summarized as such a construction of an extended Topological Quantum Field Theory as a symmetric monoidal 2-functor from a certain bicategory of double cobordisms into 2-vector spaces. It factorizes as $Z_G(-) = \Lambda \circ \mathcal{A}_0(-)_G$. The “2-linearization” functor Λ , and reproduces the *untwisted* Dijkgraaf-Witten model, in the general case of surfaces with boundary, as described by Freed and Quinn [17].

In particular, to reproduce the Dijkgraaf-Witten model, we consider only the situation down to codimension 2, unlike the full program of [16]. That is, top-dimension cobordisms are 2-morphisms in the cobordism category $\mathbf{3Cob}_2$, and objects are codimension-2 manifolds. However, $\mathbf{3Cob}_2$ is a monoidal 2-category, so we may understand it as a 3-category with one object (or isomorphism class of

objects), which we understand as “the point”. This would then be assigned the simplest 3-vector space, $\mathbf{2Vect}$, and our 2-vector spaces would be 3-linear maps interpreted as 1×1 matrices. We will not emphasize this point of view, though.

Rather, for the construction we describe here, the “quantization functor”, at least initially, is the 2-linearization process [36], which is a canonical construction giving a (symmetric monoidal) 2-functor Λ . It generalizes “linearization” of sets: the process of taking sets to the free vector space on them, and spans of sets to linear maps described by matrices. These matrices count the elements over a given pair of elements. Interpreting a set as a collection of states, and a span of sets as a collection of “histories”, this process can be seen as a very elementary sort of “geometric” quantization via sum-over histories.

The 2-linearization process categorifies this construction, and comes closer to a physically realistic quantization functor. Here we replace function spaces on sets with functor categories of groupoids (i.e. representation categories). Thus, we interpret it as a kind of “categorified quantization” of the category of groupoids and spans. It also proceeds by a sum-over-histories, or rather a colimit-over-histories. It produces not a Hilbert space, but a 2-Hilbert space. At the 2-morphism level, it reproduces features of more physical quantization by including weights which count symmetries.

Λ is a rather degenerate case, essentially because it is a canonical choice for quantization of $\text{Span}(\mathbf{Gpd})$. As a toy model of the physics of quantum field theory, though, this is rather inadequate. In particular, the full DW model crucially involves a “twisting” by a group-cocycle. This motivates the introduction of an analog of a Lagrangian “action functional” - a function valued in $U(1)$. The most canonical, but least general, situation is the untwisted case where the action functional takes the constant value 1. The full data, however, is supplied by the classical part of the twisted theory and can be described in the language of bundles and gerbes [45]. We will describe a modification of the quantization functor, $\Lambda^{U(1)}$, which accepts groupoids with some extra data.

The general DW model is “twisted” by a group cocycle ω , which gives a nontrivial topological action term. Since the fields are determined by holonomies, which are group elements, the action must be some function of group elements. The cocycle condition arises because of just the consistency conditions on ω which are needed to define $\Lambda^{U(1)}$.

The “classical theory” part of the construction here is based on topological gauge theory: that is, the theory of *flat* connections on manifolds. We adopt the view that the space of connections on a manifold should be understood as a groupoid, in which the objects are connections and the morphisms are gauge transformations. That is, the groupoid describes the local symmetries of the configuration space of the gauge theory. For the case of Lie groups, it would be useful to understand this groupoid as representing a *stack*, but in the present, discrete setting, this is more sophistication than we need.

Then, space of connections on a cobordism between two manifolds is represented as a span of groupoids (as described in section 3.2). This provides the data on which Λ will act to give our ETQFT. In particular, the configuration spaces for our flat gauge theory, denoted $\mathcal{A}_0(M)_G$, are 1-groupoids, so this is a groupoid-valued field theory.

The plan of the paper is as follows: in Section 2 we recall the categorical setup of extended topological quantum field theories; in Section 3 we describe the untwisted form of the construction using 2-linearization as the quantization functor and show that it reproduces the untwisted Dijkgraaf-Witten model; in Section 4 we give some calculations to explicitly show some of the invariants computed by this process; in Section 5 we describe the cocycle-twisted variations of the gauge theory and the 2-linearization functor and show that these reproduce the twisted Dijkgraaf-Witten model; finally we offer some concluding remarks.

2. TOPOLOGICAL QUANTUM FIELD THEORIES

Here we summarize the context of TQFT and ETQFT in which we will be working, in categorical language. We will assume in the following a familiarity at least with 2-categories here, and refer the reader to works such as those by Cheng and Lauda [10], or Lack [28]. For a good introduction on higher category theory in the context of TQFTs, see work of Baez and Dolan [4].

A topological quantum field theory (TQFT) is understood physically as a quantum field theory with *no local degrees of freedom*. In particular, we are interested in TQFTs given by gauge theories. Fields in gauge theory are connections on bundles over some base space. By assuming such connections are invariant under one-parameter families of diffeomorphisms, we find that the connections are *flat*. Thus, they are locally trivial, and the only interesting information about them is given by holonomies around non-contractible loops.

2.1. The Category \mathbf{nCob} . In general, a TQFT can be described as a functor from a category of manifolds and cobordisms into vector spaces (or Hilbert spaces) and linear maps. A survey of categorical aspects of TQFT was given by Bartlett [7]. For our purposes, we first need to understand the category of cobordisms involved here.

A *cobordism* between manifolds S_1, S_2 is a compact manifold with boundary, M , with ∂M isomorphic to the disjoint union $S_1 \amalg S_2$. Cobordisms are composed by identifying their boundaries. For our purposes, it will be useful to think of cobordisms as special kinds of *cospan*s in a category of manifolds with boundary, so that this is a special case of the composition of cospan by pushout. Our aim here is to describe a generalization of categories of cobordisms.

In [35] there is a definition of a bicategory \mathbf{nCob}_2 , which is a sub-bicategory of the span bicategory of $\mathbf{ManCorn}$, the category of manifolds with corners, and has:

- **Objects** $(n - 2)$ -manifolds P (without boundary)
- **Morphisms:** cobordisms $P_1 \times I \xrightarrow{i_1} S \xleftarrow{i_2} P_2 \times I$ where S is an $(n - 1)$ -dimensional collared cobordism with corners
- The 2-morphisms of \mathbf{nCob}_2 are generated by:
 - diffeomorphisms of cobordisms fixing the collar.
 - n -dimensional collared cobordisms with corners M , up to diffeomorphisms which preserve the boundary

Composition is by gluing along collars. (The need for collars is to ensure that there is a smooth structure on composites).

Actually, the definition given there (as a “double bicategory”) is slightly trickier, but this will suffice for our purposes. A related construction is given by Grandis [19], following from a general framework with cospan and cubical n -categories [18]. This

allows our construction here to generalize to higher n -categories. It is shown in [35] how a bicategory can be obtained from the double bicategory described there. The bicategory obtained in this way is equivalent to the usual bicategory of cobordisms between cobordisms, in which the boundaries are fixed, so our choice of the cubical framework is mainly of interest because it allows us to think of cobordisms with corners as “evolution of manifolds with boundary” in which the boundary need not be fixed. However, the category of cobordisms is not greatly different from the usual setup.

Moreover, details about smooth structure can be largely passed over here. Indeed, for our construction to work, it is only necessary to get an n -functor from the cobordism category into the appropriate form of $Span(Gpd)$. Since this is done, here by passing through the fundamental groupoid, only the homotopy type of the manifolds and cobordisms is detected by these invariants. Thus, the precise details of composing cobordisms with collars is not crucial for these ETQFTs based on gauge theory. It may be relevant for other field theories, however. TQFT is a special situation, which we now recall.

2.2. TQFT and ETQFT as Functors. Atiyah’s axiomatic formulation [2] of the axioms for a TQFT can be summarized as follows:

Definition 1. *An n -dimensional Topological Quantum Field Theory is a (symmetric) monoidal functor*

$$(3) \quad Z : \mathbf{nCob} \rightarrow \mathbf{Vect}$$

where \mathbf{nCob} is the monoidal category of $(n-1)$ -dimensional manifolds and n -dimensional cobordisms, and \mathbf{Vect} is the monoidal category whose objects are vector spaces, whose arrows are linear transformations, and whose monoidal operation is the usual tensor product \otimes .

A more general characterization of cobordism categories is the Baez-Dolan Cobordism Hypothesis, characterizing the n -category whose objects are points and whose n -morphisms are n -dimensional cobordisms (necessarily with corners). The characterization is that this category is a free symmetric monoidal n -category “with duals” in a suitable sense (details can be found in [4]). This has been proved by Lurie (see [31]). Christopher Schommer-Pries has given a presentation for $\mathbf{2Cob}_2$ as a symmetric monoidal bicategory [40], given in terms of Morse theory and a classification of singularities.

This takes us to ETQFTs, which are defined not just on manifolds with boundary, but also on manifolds with corners. This idea was introduced by Ruth Lawrence [30], under the name “ r -ETFT”, replacing the concept of vector space with that of r -vector space. Just as a TQFT assigns a space of states to a manifold and a linear map to a cobordism, a (doubly) extended TQFT will assign some such algebraic data to manifolds of dimension $(n-r)$, and cobordisms up to dimension n . Our construction here will describe the situation where $r=2$.

Recall the definition of 2-vector space (a slight abstraction of that given in [24]):

Definition 2. *The 2-category $\mathbf{2Vect}$ has:*

- **Objects:** *Finite-dimensional Kapranov-Voevodsky 2-vector spaces (i.e. \mathbb{C} -linear, finitely semisimple abelian categories)*
- **Morphisms:** *2-linear maps (i.e. \mathbb{C} -linear functors, which are necessarily additive)*

- **2-Morphisms:** *Natural transformations*

So then a straightforward categorification of Atiyah’s description of a TQFT as a functor, as proposed by Lawrence, runs as follows:

Definition 3. *An extended TQFT is a (symmetric monoidal) weak 2-functor*

$$(4) \quad Z : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$$

In particular, such a Z assigns:

- To an $(n - 2)$ -manifold, a 2-vector space
- To an $(n - 1)$ -manifold, a 2-linear map between 2-vector spaces
- To an n -manifold, a 2-natural transformation between 2-linear maps

All this data satisfies the conditions for a weak 2-functor (e.g. it preserves composition and units up to coherent isomorphism, and so forth). The monoidal structure in $\mathbf{2Vect}$ is the Deligne tensor product on abelian categories (see e.g. section 4.3 of [27]).

As before, a fuller version of this theory will use $\mathbf{2Hilb}$ (see [3]) in place of $\mathbf{2Vect}$, but we will mostly omit this complication here.

2.3. Topological Gauge Theory and TQFTs. Quantum field theories are often described in terms of a Hilbert space which is a space of sections of a G -bundle - that is, in terms of gauge theory. So it should not be surprising that TQFTs, and ETQFTs, should turn out to be related to *topological* gauge theory. This deals with flat connections on a manifold B . Being flat, the nontrivial information about a connection depends only on the topology of B .

In particular, all the information available about any connection comes in the form of holonomies of the connection around loops. The holonomy is an element $A(\gamma) \in G$ associated to a loop γ in B , defining the “parallel transport” around that loop. The G -connection is flat exactly if the holonomy assigned to a loop depends only on the homotopy class of γ . To say this more conveniently, we first recall the definition (see Brown [9]):

Definition 4. *The fundamental groupoid $\Pi_1(B)$ of a space B is a groupoid with points of B as its objects, and whose morphisms from x to y are just all homotopy classes paths in B starting at x and ending at y .*

Then suppose G is a group, understood as a one-object groupoid whose composition is the group multiplication. Then we have:

Definition 5. *A flat G -connection is a functor*

$$(5) \quad A : \Pi_1(B) \rightarrow G$$

A *gauge transformation* $\alpha : A \rightarrow A'$ from one connection to another is a natural transformation of functors so that $\alpha_x \in G$ satisfies $\alpha_y A(\gamma) = A'(\gamma) \alpha_x$ for each path $\gamma : x \rightarrow y$. Flat connections and natural transformations form the objects and morphisms of the category

$$(6) \quad \mathcal{A}_0(B)_G = [\Pi_1(B), G]$$

(Here we are using the notation that $[C_1, C_2]$ is the category whose objects are functors from C_1 to C_2 and whose morphisms are natural transformations.)

This definition of the groupoid of flat connections in terms of holonomy functors evidently is not exactly the same as the usual definition in terms of bundles, since

not all bundles are trivial. The groupoid of connections, however is equivalent, in the categorical sense, to the usual definitions in terms of principal G -bundles, as was established by Thurston [42]. This is because, as categories, $G \simeq \mathbf{G-Tor}$, the category of G -torsors, even though $\mathbf{G-Tor}$ has a much larger set of objects. A flat connection on a principal G -bundle gives the fiber-selecting functor from $\Pi_1(B)$ to $G - Tor$, where the holonomy along a path transports each fiber.

Remark 1. We note that, more precisely, the configuration spaces should be seen as stacks, which are determined by Morita equivalence classes of groupoids, and consequently everything we construct here is unchanged, up to equivalence, by taking Morita equivalent groupoids everywhere. This refinement is important for topological groupoids, but here we need not be concerned with it, since Morita equivalence and categorical equivalence are the same for finite groupoids. The point is that the groupoid of holonomy functors is *equivalent* to the groupoid of flat connections, and any representative of this equivalence class will do for our needs.

In gauge theory, two connections which are related by a gauge transformation describe physically indistinguishable states - the differences they detect are due only to the system of measurement used. The groupoid then describes the internal symmetry of a “physical” space of states. Now, if $\gamma : x \rightarrow x$ in $\Pi_1(B)$ is a loop, and A and A' are two connections related by a gauge transformation α , we have $A'(\gamma) = \alpha(x)^{-1}A(\gamma)\alpha(x)$ - that is, the holonomies assigned by the two connections around the loop are conjugate. So physically distinct holonomies correspond to conjugacy classes in G .

Indeed any category is equivalent, as a category, to its skeleton, so in general $\Pi_1(B) \cong \coprod_{b \in \pi_0(B)} \pi_1(B, b)$. The gauge transformations for connections measured from a fixed base point b are determined by a single group element at b , acting on holonomies around any loop by conjugation. The groupoid $\mathcal{A}_0(B)_G$, which is the configuration space for our theory, is then the “weak” or “stack” quotient of the space of connections by the action of gauge transformations at each base-point. We will refer to it as “the groupoid of connections”.

Proposition 1. *For any compact manifold B , and finite group G , the groupoid $\mathcal{A}_0(B)_G$ is essentially finite.*

Proof. First, note that for any space B ,

$$(7) \quad \Pi_1(B) \cong \coprod_{i=1}^n (\pi_1(B_i))$$

The sum is taken over all path components of B . That is, objects in $\Pi_1(B)$ are by definition isomorphic if and only if they are in the same path component. The automorphisms for an object corresponding to path component B_i are the equivalence classes of paths from any chosen point to itself—namely, $\pi_1(B_i)$. If B is a compact manifold, so is each component B_i , which is also connected. But the fundamental group for a compact, connected manifold is finitely generated. So in particular, each $\pi_1(B_i)$ is finitely generated, and there are a finite number of components. So $\Pi_1(B)$ is an essentially finitely generated groupoid.

But if $\Pi_1(B)$ is essentially finitely generated, then since G is a finite group, $\mathcal{A}_0(B)_G$ is an essentially finite groupoid. This is because each functor’s object map is determined by the images of the generators, and there are finitely many

such assignments. Similarly, $\Pi_1(B)$ is equivalent to its skeleton, and a natural transformation in this case is just given by a group element in G for each component of B , so there are finitely many. \square

We have described how to associate the groupoid $\mathcal{A}_0(B)_G$ to any manifold B

3. ETQFT BY 2-LINEARIZATION

Here we want to consider explicit construction of some extended TQFTs based on a finite group G , using the gauge groupoids described in the previous section.

In [36], we defined a weak 2-functor Λ from spans of groupoids to 2-vector spaces. In particular, the construction we give here works by associating spans of groupoids to cobordisms, and then applying this Λ . These groupoids arise from the the moduli space of flat connections on the source and target manifolds, as well as on the cobordism itself. These are connected by natural restriction maps to give spans. A similar line of reasoning gives spans of span maps associated to cobordisms between cobordisms.

We have noted moduli spaces should be seen as groupoids because there is an action of the gauge group, by means of gauge transformations. Strictly speaking, our constructions are all invariant, up to categorical equivalence, under Morita equivalence of groupoids. So in fact, the moduli spaces should be seen as the stacks (Morita equivalence classes) represented by these groupoids. This distinction would be more important in the case of topological or Lie groupoids, however, and will mostly be omitted here.

We recap the key ingredient Λ next.

3.1. Groupoidification and 2-Linearization. *Groupoidification* is a (non-systematic) process which reverses the (systematic) “degroupoidification” functor, which gives representations of $\text{Span}(\mathbf{Gpd})$ in \mathbf{Vect} , or \mathbf{Hilb} . The goal is to find structures in $\text{Span}(\mathbf{Gpd})$ whose representations reproduce some chosen structure in \mathbf{Vect} or \mathbf{Hilb} . The reader may find more details on this program in a review by Baez, Hoffnung and Walker [5], and in Hoffnung’s work on geometric representation theory [22]. The author has described an example of an application to physics, and in particular the combinatorics of Feynman diagrams in [34].

2-Linearization provides an invariant Λ for $\text{Span}(\mathbf{Gpd})$ which is analogous to the functor in degroupoidification. This is discussed in the general setting in [36]. This has been discussed as a setting for low-dimensional topology by Yetter [46]

Both invariants rely on different forms of a ubiquitous **pull-push** process whose best-known example is found in ordinary matrix multiplication. This is the form which appears in groupoidification. In the context of the 2-linearization functor Λ , the “pull” and “push” refer to the direct and inverse limits of \mathbf{Vect} -presheaves along a functor. This fundamental construction appears in many categories, notably toposes [32]. For abelian sheaves this is described in some generality by Kashiwara and Schapira [25]. Here we mention only a few examples closely related to the present case, namely in the setting of representations of rings [8].

For our immediate purposes, we can omit many of these considerations, but note that the *ambidextrous* adjunction between direct and inverse image functors valued in \mathbf{Vect} gives us the extra structure used to construct Λ . This ambidextrous adjunction appears, indirectly, because a finite-dimensional vector space is canonically isomorphic to its double-dual. (For this reason, in infinite-dimensional situations,

one properly ought to use **Hilb**-valued functors, which may be seen as equivariant Hilbert bundles. For the finite case, **Vect** is sufficient.)

3.2. The 2-Linearization Functor Λ . The category of representations $Rep(X)$ for a groupoid X consists of a category of X -actions on bundles of vector spaces over the objects of X . This is just the category of functors from X into **Vect**, which we denote $[X, \mathbf{Vect}]$. (We work here just with groupoids in **Set**: small categories with invertible morphisms.) Such a representation category is a 2-vector space (**Vect**-enriched abelian category), which is the kind of structure which an Extended TQFT in n dimensions assigns to a manifold of dimension $(n - 2)$.

The basic point of Λ is to use the fact that, for any functor $f : X \rightarrow Y$ of essentially finite groupoids (those which are equivalent as categories to finite groupoids), there is a two-sided adjunction

$$(8) \quad Rep(X) \begin{array}{c} \xleftarrow{f_*} \\ \xrightarrow{f^*} \end{array} Rep(Y)$$

where $Rep(X)$ is the category of (finite dimensional) representations of X .

We describe these as f^* (“pull”) and f_* (“push”) between the 2-vector spaces of functors $[X, \mathbf{Vect}]$ and $[Y, \mathbf{Vect}]$. In fact, this adjunction is ambidextrous: f_* is both a left and a right adjoint to f^* . The importance of such two-sided, or “ambidextrous” adjunctions is discussed by Lauda [29] from the algebraic point of view which relates 2D TQFTs to Frobenius algebras.

The effect of Λ on 2-morphisms can also be thought of in terms of a “pull-push” process, but here we use the unit and counit from the two adjunctions between f^* and f_* . In particular, we use the unit from the adjunction where f_* is a right adjoint, and the counit from the adjunction where it is a left adjoint. We denote the unit:

$$(9) \quad \eta_R : \text{Id}_{[X, \mathbf{Vect}]} \Longrightarrow f_* f^*$$

The counit is similarly denoted:

$$(10) \quad \epsilon_L : f_* f^* \Longrightarrow \text{Id}_{[X, \mathbf{Vect}]}$$

We note here that these two operations are a special case of the general “six-operation” framework [33]: in algebraic geometry, for a map $f : X \rightarrow Y$ of varieties (or schemes), one gets functors f^* , f_* , $f^!$ and $f_!$ between categories of sheaves $Sh(X)$ and $Sh(Y)$. This is a special case, since we take our groupoids to have discrete topology, so all functors (as presheaves) are sheaves. Furthermore, f^* has in the general case a different left adjoint f_* and right adjoint $f_!$. However, in this case, two pairs of functors coincide (which is due, indirectly, to the fact that objects in **Vect** are canonically isomorphic to their double-duals, which can be seen by the matrix representation of these 2-linear maps).

One way to summarize the structure we get uses a 2-category of *spans of groupoids*:

Definition 6. *The symmetric monoidal 2-category $\text{Span}(\mathbf{Gpd})$ has:*

- **Objects:** *Essentially finite groupoids*
- **Morphisms:** *Spans of groupoids*
- **2-Morphisms:** *Isomorphism classes of spans of span maps*

The monoidal operation is the product in \mathbf{Gpd} .

This generalizes a construction of a 2-category whose morphisms are spans, and whose 2-morphisms are span maps. In fact, $\text{Span}(\mathbf{Gpd})$ as we have presented it might be better understood a 3-category. In general, the span construction on any bicategory will yield a (monoidal) tricategory, where the 3-morphisms are maps of spans of span maps, as described Hoffnung [23]. Reducing to 3-isomorphism classes gives exactly the 2-morphisms described here, and makes our $\text{Span}(\mathbf{Gpd})$ a monoidal 2-category. We have chosen the current approach because of the up-to-diffeomorphism definition of 2-morphisms in \mathbf{nCob}_2 .

For a category \mathbf{C} with pullbacks, the $\text{Span}(\mathbf{C})$ has many useful properties due to certain universal properties of the span construction [12] (for bicategories, similar analysis is done in [26]). For example, taking categories of spans ensures that every morphism has a “dual” (the same span, considered with the opposite orientation), and is a minimal expansion of \mathbf{C} with this property. The point of the following construction is to take these “duals” and represent them as ambi-adjoint functors.

Thus, it was shown [36] that the following defines a 2-functor:

Definition 7. *The weak 2-functor*

$$(11) \quad \Lambda : \text{Span}(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

assigns:

- For X an essentially finite groupoid, the functor category $\Lambda X = [X, \mathbf{Vect}]$
- For a span of groupoids $A \xleftarrow{s} X \xrightarrow{t} B$ in $\text{Span}(\mathbf{Gpd})$, the 2-linear map:

$$(12) \quad \Lambda X = t_* \circ s^* : \Lambda A \longrightarrow \Lambda B$$

- For a span between spans, $Y : X_1 \rightarrow X_2$ for $X_1, X_2 : A \rightarrow B$, as in:

$$(13) \quad \begin{array}{ccccc} & & X_1 & & \\ & s_1 \swarrow & \uparrow s & \searrow t_1 & \\ A & & Y & & B \\ & \nwarrow s_2 & \downarrow t & \nearrow t_2 & \\ & & X_2 & & \end{array}$$

the natural transformation

$$(14) \quad \Lambda(Y) = \epsilon_{L,t} \circ N \circ \eta_{R,s} : (t_1)_* s_1^* \Longrightarrow (t_2)_* s_2^*$$

where $\epsilon_{L,t}$ is the counit for the left adjunction associated to t , and $\eta_{R,s}$ is the unit for the right adjunction associated to s , and N is the Nakayama isomorphism between the left and right adjoints.

We note that Λ is a weak 2-functor, so there are also natural isomorphisms called the “compositor”

$$(15) \quad \beta : \Lambda(X' \circ X) \rightarrow \Lambda(X') \circ \Lambda(X)$$

for each composable pair of spans X and X' , and the “unitors”

$$(16) \quad U_B : 1_{\Lambda(B)} \xrightarrow{\sim} \Lambda(1_B)$$

for each groupoid X . These are described in [36] in detail. So, briefly, is the case where the 2-morphism diagram is only required to commute up to isomorphism. These issues will not be required in the current context.

The role of the Nakayama isomorphism here is also described in more detail in [36], but is relevant here, so we will briefly recap this. In general, given a map $f : X \rightarrow Y$, there will be both a left and a right adjoint to f^* , the pullback of (in this case, **Vect**-valued) functors from Y to X . These may be described in terms of the internal hom and \otimes in **Vect**.

In each case, these “pushforwards” of a functor $F : X \rightarrow \mathbf{Vect}$ to Y will be described as a direct sum over all objects x in the essential preimage of $y \in Y$. Since $F(x)$ gives a representation of $Aut(x)$, the summands are the induced representations along the associated homomorphism $\hat{F} : Aut(x) \rightarrow Aut(y)$. For the left adjoint, this is $\mathbb{C}[Aut(y)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$ (a representation of $Aut(y)$), and for the right adjoint it is $\text{hom}_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(y)], F(x))$ (that is, the hom-space as $\mathbb{C}[Aut(x)]$ -modules). The Nakayama isomorphism turns a map in the right adjoint (hom-space) to a vector in the left adjoint (tensor product) by the “exterior trace”, averaging over a group action:

$$(17) \quad \phi \mapsto \frac{1}{\#Aut(x)} \sum_{g \in Aut(y)} g^{-1} \otimes \phi(g)$$

This gives the natural transformations associated to 2-morphisms by 2-linearization. Note that we sum over $Aut(y)$, which projects to the $Aut(y)$ -invariant subspace, a canonical representative of a vector in the tensor product, but the “average” is given by dividing by the size of $Aut(x)$. This reflects the fact that we are pushing forward a representation of $Aut(x)$, and is necessary to make this an isomorphism when we are dealing with modules in general, say over \mathbb{Z} , rather than \mathbb{C} -vector spaces. In this setting, it merely sets a canonical scale, which turns out to reproduce the groupoid cardinality which appears in the groupoidification process of Baez and Dolan (see e.g. [6]).

The 2-linearization construction relies on the fact that having both covariant and contravariant functors $(-)^*$ and $(-)_*$ amounts to the same thing as having a single functor from $\text{Span}(\mathbf{Gpd})$. In general, pairs of functors like this satisfying some nice properties are *Mackey functors* (see [37, 20]). The situation is in general somewhat more complicated when groupoids are thought of as having *spaces*, rather than discrete sets, of objects and morphisms, whose topology is nontrivial. However, we take advantage of the simplifying fact for the discrete case to construct an ETQFT for a discrete gauge group G . We describe this in the next section.

3.3. From Cobordisms to Spans. In this section, we show the functoriality of the second half of the factorization $Z_G = \Lambda \circ \mathcal{A}_0(-)_G$.

Theorem 1. *There is a 2-functor:*

$$(18) \quad \mathcal{A}_0(-)_G : \mathbf{nCob}_2 \rightarrow \text{Span}(\mathbf{Gpd})$$

induced by the effect on objects.

Proof. To begin with, we note that this is true at the level of objects. The functor category $\mathcal{A}_0(B)_G$ is a groupoid, since any natural transformation g assigns to a point $b \in B$ a group element g_b , which is invertible. The transformation g^{-1} with $g_b^{-1} = (g_b)^{-1}$ is the inverse.

A cobordism in \mathbf{nCob}_2 can be seen as a particular cospan in $\mathbf{ManCorn}$, given by inclusion maps:

$$(19) \quad \begin{array}{ccc} & S & \\ \iota \nearrow & & \nwarrow \iota' \\ B & & B' \end{array}$$

(Note that as described in [35], there are, moreover, “collars” associated with these inclusions, but these have no effect on our construction so we shall ignore them here.)

Our construction amounts to a sequence of functorial operations, which therefore give a corresponding sequence of spans (or cospans) in three different categories. The first step is the operation of taking the fundamental groupoid.

Given two cobordisms $S : B_1 \rightarrow B_2$ and $S' : B_2 \rightarrow B_3$, the composite $S' \circ S$ is a (homotopy) pushout of two cospans (over B_2). The functor Π_1 also gives cospans of the fundamental groupoids, whose composite $\Pi_1(S') \circ \Pi_1(S)$ is a (weak) pushout. Then it is a well-known consequence of the Brown’s [9] groupoid version of the Van Kampen theorem that $\Pi_1(S' \circ S) \simeq \Pi_1(S') \circ \Pi_1(S)$ (see also [21]).

In the next step, we apply $[-, G] : \mathbf{Cosp}(\mathbf{Gpd}) \rightarrow \mathbf{Span}(\mathbf{Gpd})$. So at this stage of the construction we have a span. To see that this operation is compatible with composition of cobordisms the essential fact is that the contravariant functor $[-, G]$ takes weak (homotopy) pushouts to weak pullbacks.

By construction of \mathbf{nCob}_2 , the composition of morphisms is by weak pushout of (collared) cospans in $\mathbf{ManCorn}$. This still holds when we take fundamental groupoids. Applying $[-, G]$, we get spans of groupoids. Thus the corresponding diagram of spans contains a (weak) pullback square. Denoting the pull-backs along the inclusions by $\iota^* = \pi$ and $\iota'^* = \pi'$, we have this diagram:

$$(20) \quad \begin{array}{ccccc} & & \mathcal{A}_0(S')_G \circ \mathcal{A}_0(S)_G & & \\ & \swarrow^{p_1 \circ P_S} & & \searrow^{p_{S'}} & \\ & \mathcal{A}_0(S)_G & \xrightarrow[\sim]{\alpha} & \mathcal{A}_0(S')_G & \\ & \swarrow^{p_1} & & \searrow^{p'_1} & \\ \mathcal{A}_0(B_1)_G & & \mathcal{A}_0(B_2)_G & & \mathcal{A}_0(B_3)_G \end{array}$$

The weak pullback is canonically described (up to equivalence) as a comma category, whose objects are triples A, f, A' where $A \in \mathcal{A}_0(S)_G$, $A' \in \mathcal{A}_0(S')_G$, and $f : p_2(A) \rightarrow p'_1(A')$. That is, connections A and A' on S and S' , the restrictions to B_2 may be gauge equivalent, but not necessarily equal. Each different gauge equivalence gives a different object of $\mathcal{A}_0(S')_G \circ \mathcal{A}_0(S)_G$.

Thus, $\mathcal{A}_0(S')_G \circ \mathcal{A}_0(S)_G \simeq \mathcal{A}_0(S' \circ S)_G$, where composition of cobordisms is by the (homotopy) pushout along the collared inclusions of the boundary B_2 .

A similar argument shows the unitor property for $\mathcal{A}_0(-)_G$.

Finally, for 2-morphisms, we note that here, composition is by strict pullback and pushout since spans of spans are taken only up to isomorphism. Otherwise the same argument holds. Thus, we have a 2-functor into $\mathbf{Span}(\mathbf{Gpd})$. \square

In our example, the connections on S and S' need only restrict to gauge-equivalent connections on B_2 —since two such connections can be “pasted” together using a gauge transformation. Moreover, since all these categories are groupoids, every morphism mentioned is invertible.

Remark 2. We also note here that a similar construction to the functor $[-, G]$ used here plays a roll in a construction by Grandis [19] of TQFT via spans of sets. In that case, Grandis uses that this property holds for contravariant functors $[-, S] : \mathbf{Top} \rightarrow \mathbf{Set}$. In [19], this is done for topological spaces, and $[-, S]$ is the functor which takes the set of homotopy classes of maps into S , rather than the functor category as in our notation here. That is, we are now concerned with a homotopy 1-type (a groupoid), rather than a 0-type (a set) of the space of maps into S . The result we need is shown in the general case of spaces by Chorny [11], and the groupoid case follows since groupoids are homotopy 1-types of spaces.

3.4. Extended TQFT via Λ . We can now describe explicitly how our ETQFT is constructed:

Definition 8. For any finite group G , define the 2-functor

$$(21) \quad Z_G = \Lambda \circ \mathcal{A}_0(-)_G : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$$

Corollary 1. This Z_G is a 2-functor.

Proof. Since both $\mathcal{A}_0(-)_G$ and Λ are 2-functors [36], so is the composite Z_G , so this is indeed an Extended TQFT. \square

To summarize: for any finite group G , we now have a (weak) 2-functor $Z_G = \Lambda \circ \mathcal{A}_0(-)_G : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$ - that is, an ETQFT. To a compact $(n-2)$ -manifold, Z_G assigns a 2-vector space. This consists of representations of the groupoid of G -connections on B weakly modulo gauge transformations. The spans of groupoids associated to cobordisms will then give rise to 2-linear maps between representation categories, and spans of spans associated to cobordisms between cobordisms will give natural transformations.

In particular, since by Theorem 2.3 $\mathcal{A}_0(B)_G$ is an essentially finite groupoid for a manifold with finitely many connected components with finite fundamental groups, then it is essentially finite, so the main theorem of [36] then implies $[\mathcal{A}_0(B)_G, \mathbf{Vect}]$ is a Kapranov-Voevodsky 2-vector space.

Remark 3. To describe it explicitly, given a finite group G , the extended TQFT Z_G makes the following assignments:

- For a closed compact manifold B , Z_G assigns the 2-vector space:

$$(22) \quad Z_G(B) = [\mathcal{A}_0(B)_G, \mathbf{Vect}]$$

- For a cobordism between manifolds:

$$(23) \quad B \xrightarrow{i} S \xleftarrow{i'} B'$$

the weak 2-functor assigns a 2-linear map:

$$(24) \quad Z_G(S) = (p')_* \circ p^*$$

where p and p' are the projections for the associated groupoids.

- For a cobordism with corners between two cobordisms with the same source and target:

$$(25) \quad \begin{array}{ccccc} & & S_1 & & \\ & i_1 \nearrow & \downarrow i & \nwarrow i'_1 & \\ B & & M & & B' \\ & i_2 \searrow & \uparrow i' & \swarrow i'_2 & \\ & & S_2 & & \end{array}$$

the natural transformation (14) becomes:

$$(26) \quad \Lambda(M) = \epsilon_{L,(p')} \circ N \circ \eta_{R,p} : (p'_1)_* \circ p_1^* \implies (p'_2)_* \circ p_2^*$$

where p' and p are as above. The coherence isomorphisms which make Z_G a weak 2-functor are those defined by Λ as in [36]. (These appear in coordinates as matrices whose components are linear maps between the coefficients of the 2-linear maps).

This is a fairly straightforward result, giving a categorical account of a construction which has already been studied extensively. We will next see how these constructions appear in the language described here.

3.5. The 3D Untwisted Dijkgraaf-Witten Model as ETQFT. Now we consider the case of $\mathbf{3Cob}_2$, and another special case of this ETQFT construction. Given a gauge group G , the Dijkgraaf-Witten (DW) model [13] is a topological gauge theory, involving flat G -connections on manifolds. For Lie groups, this theory is related to the Chern-Simons theory, but our interest here is for finite groups. The general theory of Lie groups can be understood from finite groups and simply connected Lie groups. This is because, as described by [13], the finite groups occur in exact sequences as either the group of components, or the fundamental group, of Lie groups (which can thus be used to reduce a general Lie group first to a connected, then a simply-connected, one).

TQFTs equivalent to the DW model are often defined as invariants for triangulated manifolds, found by considering compatible G -colorings of the (directed) edges. This is done, for example, by Yetter [47], and in discussion in the chapter on TQFT of the unpublished notes by Porter [39], which also discuss an extension to categorical groups. Yetter showed that one can obtain an invariant of manifolds which is independent of triangulation via a colimit over all triangulations.

While triangulations are crucial in the case of categorical groups, for ordinary groups, the G -colorings of edges in a triangulation amount to flat G -connections. These can be described in terms of maps in $\text{Hom}(\pi_1(M), G)$, or equivalently in $\text{Hom}(M, BG)$, where BG is the classifying space of G . We use the former description here, since the groupoid structure is easiest to see in that form.

We remark that the normalizing factors which appear in the 2-functor Λ as the Nakayama isomorphism (or just groupoid cardinality) count the size of automorphism groups of objects. In the mapping space $\text{Hom}(M, BG)$, these appear as the size of homotopy groups of connected components (as in the ‘‘homotopy order’’ as described in [39]). The homotopy order of a connected space X with base-point x

and only finitely many nontrivial homotopy groups is:

$$(27) \quad \#\pi(X, x) = \prod_{i=1}^{\infty} |\pi_i(X, x)^{-1^i}|$$

(In the case of BG , and $Hom(M, BG)$, all homotopy groups for $i \geq 2$ are trivial, so this reproduces the groupoid cardinality.)

We now consider explicitly how the DW model and a natural ETQFT extension of it can be found from our Z_G .

Recall that the category $\mathbf{2Cob}$ occurs in $\mathbf{3Cob}_2$ as the category of automorphisms of the object \emptyset , which is particularly interesting since \emptyset is the monoidal unit in $\mathbf{3Cob}_2$. We can ask about the effect of Z_G restricted to this automorphism category. It turns out to be just the same as the DW model in 3 dimensions.

Our main theorem is the following:

Theorem 2. *There is a natural isomorphism between the functor $Z_G|_{Aut(\emptyset)}$ and the untwisted DW model.*

Proof. We need to exhibit the natural equivalence at the level of objects and morphisms.

Now, $Z_G(\emptyset) \cong \mathbf{Vect}$, whose single basis 2-vector (mapped to \mathbb{C} under the equivalence) is the trivial representation of the trivial group.

So a cobordism in $Aut(\emptyset)$ gives a 2-linear map from \mathbf{Vect} to \mathbf{Vect} , which is naturally equivalent to giving a vector space (and in particular, a complex vector space with a specified basis, and thus a Hilbert space). Cobordisms in $Aut(\emptyset)$ are 2-dimensional cobordisms from \emptyset to \emptyset , or in other words, closed 2-dimensional manifolds. These are, up to diffeomorphism, just genus- g tori Σ_g .

Given Σ_g , the DW model produces a d -dimensional Hilbert space \mathcal{H}_{Σ_g} , where

$$(28) \quad \begin{aligned} d &= |V_g| \\ &= |\text{hom}(\pi_1(\Sigma_g), G)/G| \end{aligned}$$

with a basis canonically indexed by conjugacy classes of flat connections $\gamma \in \text{hom}(\pi_1(\Sigma_g), G)/G$.

Now, thinking of Σ_g as a cobordism, that is, as a cospan:

$$(29) \quad \emptyset \rightarrow \Sigma_g \leftarrow \emptyset$$

we get the span of groupoids

$$(30) \quad \mathbf{1} \xleftarrow{!} \mathcal{A}(\Sigma_g) // G \xrightarrow{!} \mathbf{1}$$

Then by the above, we find that $Z_G(\Sigma_g) : Z_G(\emptyset) \rightarrow Z_G(\emptyset)$ can be represented as a 1×1 matrix, with

$$(31) \quad \begin{aligned} Z_G(\Sigma_g)_{\mathbb{C}, \mathbb{C}} &= \langle !*(\mathbb{C}), !*(\mathbb{C}) \rangle \\ &= \bigoplus_{\gamma \in \mathcal{A}/G} \text{hom}(\mathbb{C}, \mathbb{C}) \\ &\cong \bigoplus_{\gamma \in \mathcal{A}/G} \mathbb{C} \end{aligned}$$

This is canonically isomorphic to \mathcal{H}_{Σ_g} .

So suppose we have a 3-dimensional cobordism between 2-manifolds Σ and Σ' , which amounts to a cospan of cospan maps:

$$(32) \quad \begin{array}{ccccc} & & \Sigma & & \\ & \nearrow & \downarrow & \nwarrow & \\ \emptyset & \longrightarrow & M & \longleftarrow & \emptyset \\ & \searrow & \uparrow & \swarrow & \\ & & \Sigma' & & \end{array}$$

then again there is a span of span maps (all the groupoids thought of as spans from $\mathbf{1}$ to itself by the unique map):

$$(33) \quad \Sigma \xleftarrow{\pi} M \xrightarrow{\pi'} \Sigma'$$

Now, this gives a 2-linear map $Z_G(M)$, and again there is only one entry, so we find

$$(34) \quad Z_G(M)_{\mathbb{C},\mathbb{C}} : Z_G(\Sigma)_{\mathbb{C},\mathbb{C}} \rightarrow Z_G(\Sigma')_{\mathbb{C},\mathbb{C}}$$

And in particular, we can write the components of this linear map as:

$$(35) \quad \begin{aligned} (Z_G(M)_{\mathbb{C},\mathbb{C}})_{\gamma,\gamma'} &= |(\pi, \pi')^{-1}([\gamma], [\gamma'])| \\ &= \sum_{[\gamma''] \in (\pi, \pi')^{-1}([\gamma], [\gamma'])} \frac{1}{|Aut(\gamma'')|} \\ &= \sum_{[\gamma'']} \frac{|\{\gamma'' \in [\gamma']\}|}{|G|} \\ &= \frac{1}{|G|} \sum_{\gamma''} 1 \end{aligned}$$

So this is the same linear map produced by the DW model. \square

The DW model itself is somewhat more general than what we have discussed so far. In fact, the 2-linearization framework used here constructs a particular ETQFT, which is the "untwisted" theory. Twisted DW models may be defined using a class α from the group cohomology of G . To produce twisted DW models, one must extend the 2-linearization framework to include cocycles. In the twisted Dijkgraaf-Witten model, the "topological action" associated to a given flat connection, is a unit complex number associated to that connection, determined by α .

In Section 5 we will describe how to extend this result through a generalized 2-linearization process.

4. EXAMPLE CALCULATIONS

Although the construction for an ETQFT will work in any dimension, its main features are visible in any dimension at least 2, to allow codimension-2 submanifolds. Some calculations in low dimensions illustrate the invariants produced by the ETQFT.

4.1. Z_G **On Manifolds.** We can give the dimension of the 2-vector space assigned to any manifold by counting its basis objects, which yields the following straightforward fact:

Proposition 2. *The KV 2-vector space $Z_G(B)$ for any connected manifold B has dimension:*

$$(36) \quad \sum_{[A] \in \mathcal{A}/G} |\{\text{irreps of Aut}(A)\}|$$

The sum is over equivalence classes of connections on B .

Proof. This is just a special case of the general fact about representation categories for groupoids.

The groupoid $\mathcal{A}_0(B)_G$ is equivalent to its skeleton S , whose objects are the gauge equivalence classes of connections on B , and whose morphisms are the stabilizer groups (of gauge transformations fixing any representative) at each object. Then $[S, \mathbf{Vect}] \simeq [\mathcal{A}_0(B)_G, \mathbf{Vect}]$, but $[S, \mathbf{Vect}]$ is a KV vector space, hence equivalent to some \mathbf{Vect}^m , where m is the number of non-isomorphic simple objects.

A functor $F : S \rightarrow \mathbf{Vect}$ assigns a vector space to each object $[A]$ (equivalence class of connections), carrying a representation of $\text{Aut}(A)$. Two functors giving inequivalent representations cannot have any nontrivial natural transformation between them, by Schur's lemma. On the other hand, any representation of $\text{Aut}(A)$ is a direct sum of irreducible representations. So a simple objects in $[\mathcal{A}_0(B)_G, \mathbf{Vect}]$ amount to a choice of $[A]$, and an irreducible representation of $\text{Aut}(A)$. The statement follows immediately. \square

Our construction works, in principle, for manifolds of any dimension, though computations become more involved in higher dimension as one might expect. Next we consider some particular examples.

Example 1. The 2-vector space assigned to the circle S^1 by Z_G is:

$$(37) \quad [\mathcal{A}_0(S^1)_G, \mathbf{Vect}] = \Lambda \circ \mathcal{A}_0(S^1)_G = \text{Rep}(\mathcal{A}_0(S^1)_G)$$

The fundamental group of the circle is \mathbb{Z} , and $\Pi_1(S^1)$ is thus equivalent to \mathbb{Z} as a one-object category. A functor $\star\mathbb{Z} \rightarrow G$ is thus a homomorphism, determined by $g \in G$, the image of $1 \in \mathbb{Z}$, so we denote this functor $g : \mathbb{Z} \rightarrow G$ also. A natural transformation is a conjugacy relation: $h : g \rightarrow g'$ assigns to the single object in \mathbb{Z} a morphism $h \in G$, and the naturality condition that $g'h = hg$, or simply $g' = hgh^{-1}$.

Thus, $\mathcal{A}_0(S^1)_G$ is equivalent to the groupoid whose objects are $g \in G$, and whose morphisms are conjugacy relations between elements. This is the transformation groupoid of the adjoint action of the group G on itself, also known as G “weakly modulo” G , or $G//G$. This is also the discrete form of the *loop groupoid* LG , as summarized in the account by Willerton [45]).

Finally, the 2-vector space corresponding to the circle is the category of representations of $G//G$.

$$(38) \quad Z_G(S^1) = \text{Rep}(G//G)$$

is generated by a basis of irreducible objects, the elements of which are labelled by pairs: a conjugacy class $[g]$ in G , and a representation of $\text{Aut}([g])$. All representations are direct sums of these. One can also think of an object of $Z_g(S^1)$ a G -equivariant vector bundle on G .

The skeleton of $G//G$ has as objects the *conjugacy classes* of G , and each object has $\text{Aut}(g) < G$, the stabilizer subgroup of g , which is the centralizer C_g .

The circle S^1 generates all compact 1-manifolds, so we now know in general what will be assigned to objects in the 3D ETQFT.

Let us briefly consider the case where objects are 2-dimensional manifolds (as in the 4D TQFT). We will not study the 4D ETQFT in detail, but this will illustrate that the same construction will work in that case.

Example 2. Consider the torus $T^2 = S^1 \times S^1$. We want to find

$$(39) \quad Z_G(T^2) = [\mathcal{A}_0(T^2)_G, \mathbf{Vect}]$$

The category $\mathcal{A}_0(T^2)_G$ is again the category of functor $\Pi_1(T^2) \rightarrow G$ and natural transformations. $\Pi_1(T^2)$ is equivalent to its skeleton, the fundamental group of T^2 , which is \mathbb{Z}^2 . So then functor $F \in [\mathbb{Z}^2, G]$ is then determined by a group homomorphism from \mathbb{Z}^2 to G . the functor F is determined by the images of the two generators $(1, 0)$ and $(0, 1)$. Since \mathbb{Z}^2 is abelian, the images $g_1 = F(1, 0)$ and $g_2 = F(0, 1)$ must commute.

So the objects of $\mathcal{A}_0(T^2)_G$ are indexed by commuting pairs of elements $(g_1, g_2) \in G^2$. (We note here that in the case of a topological group, this is a space of some interest in itself; see e.g. [1]. In the discrete case, this is simply a set.)

A natural transformation $g : F \rightarrow F'$ assigns to the single object \star of \mathbb{Z}^2 a morphism in G —that is, a group element h . This must satisfy the naturality condition $hF(a)h^{-1} = F'(a)$ for all a . This will be true for all a in \mathbb{Z}^2 as long as it is true for $(1, 0)$ and $(0, 1)$.

In other words, functors F and F' represented by $(g_1, g_2) \in G^2$ and $(g'_1, g'_2) \in G^2$, the natural transformations $\alpha : F \rightarrow F'$ correspond to group elements $h \in G$ which act in both components at once, so $(h^{-1}g_1h, h^{-1}g_2h) = (g'_1, g'_2)$.

So we have that the groupoid $\mathcal{A}_0(T^2)_G$ is equivalent to $A//G$, where $A = \{(g_1, g_2) \in G^2 : g_1g_2 = g_2g_1\}$, and the action of G on A comes from the action on G^2 as above.

So the 2-vector space $Z_G(T^2)$ is just the category of \mathbf{Vect} -presheaves on A , equivariant under the given action of G . This assigns a vector space to each connection (g_1, g_2) on T^2 , and an isomorphism of these vector spaces for each gauge transformation $h : (g_1, g_2) \mapsto (h^{-1}g_1h, h^{-1}g_2h)$. Equivalently (taking a skeleton of this), we could say it gives a vector space for each equivalence class $[(g_1, g_2)] \in G^2$ under simultaneous conjugation, and a representation of G on this vector space.

A similar pattern will apply to a 2-dimensional surface of genus k .

4.2. Z_G on Cobordisms. To help clarify the construction we have described, we consider some examples for particular cobordisms, and particular groups G .

Example: The Pair of Pants. Consider the “pair of pants” cobordism, or “trinion”, depicting two circles coalescing into one circle. This is a morphism $Y : S^1 \sqcup S^1 \rightarrow S^1$ in $\mathbf{2Cob}$, often also described as a three-punctured sphere, or a disk with two holes.

Applying the monoidal functor Z_G , the corresponding 2-linear map is:

$$(40) \quad Z_G(Y) : Z_G(S^1) \otimes Z_G(S^1) \rightarrow Z_G(S^1)$$

(The monoidal operation \otimes is the Deligne tensor product for Abelian categories. It is analogous to the tensor product for modules or vector spaces: $A \otimes B$ is a

representing object for bi-2-linear functors out of $A \times B$, and is dual to the internal *Hom* functor.)

This $Z_G(Y)$ can be described in terms of a matrix of vector spaces (by the classification theorem for a KV 2-vector space). The pair of pants thus defines a “multiplication” functor on $Z_G(S^1)$. In general, the bases for the two 2-vector spaces are the irreducible representations of the corresponding groupoid. Given irreps V , and W , the coefficients of the 2-linear map $Z_G(Y)$ are (by Frobenius reciprocity, as described in [36]):

$$(41) \quad Z_G(Y)_{V,W} \cong \text{hom}_{\text{Rep}(\mathcal{A}_0(Y)_G)}((p_1)^*V, (p_2)^*W)$$

That is, one pulls back the basis 2-vectors to give representations $(p_1)^*V$ and $(p_2)^*W$ of $\mathcal{A}_0(Y)_G$, the middle groupoid of the span. The coefficient is the “inner product”—the internal hom, which is the space of intertwiners between the two pulled-back representations. By Frobenius reciprocity, this amounts to counting the multiplicities of each irreducible in the target 2-vector space in the image of the chosen basis irrep in the source.

We compute some of these 2-linear maps here to illustrate.

To begin with, recall the 2-vector space on S^1 found in Example 1. It is equivalent to $\text{Rep}(G//G)$.

The groupoid of connections on $S^1 \cup S^1$ can be found using the fact that the path groupoid is just $\Pi_1(S^1) \cup \Pi_1(S^1)$, a disjoint union of two copies of the groupoid $\Pi_1(S^1) \cong \mathbb{Z}$. A functor into G then amounts to two choices $g, g' \in G$. A gauge transformation amounts to a conjugation by some $h \in G$ at *each* of the two objects (one chosen base points in each component), so:

$$(42) \quad \begin{aligned} \mathcal{A}_0(S^1 \cup S^1)_G &\cong (G \times G)//(G \times G) \\ &\cong (G//G)^2 \end{aligned}$$

where $G \times G$ acts on itself by conjugation component-wise. This is illustrated in Figure 1. The connection on Y has holonomies g and g' around the two holes. On $S^1 \cup S^1$, this restricts to a connection with holonomies g and g' respectively, and on S^1 to the product (since the outside S^1 is homotopic to the composite of the two loops).

FIGURE 1. Connection for Pants

The path groupoid is equivalent to $\pi_1(Y) = F(\gamma_1, \gamma_2)$, the free group on two generators. A functor in $[\Pi_1(Y), G]$ thus amounts to a pair of elements (g, g') , the images of the two generators. A gauge transformation amounts to conjugation at the single object (a chosen base point in Y). So we have the groupoid:

$$(43) \quad \mathcal{A}_0(S)_G \cong (G \times G) // G$$

in which G acts on $G \times G$ by conjugation in both components at once. Thus we have the span:

$$(44) \quad \begin{array}{ccc} & (G \times G) // G & \\ p_1 \swarrow & & \searrow p_2 \\ (G // G)^2 & & G // G \end{array}$$

Both projections are restrictions of a connection on Y to the corresponding connection on the components of the boundary. It is clear from the figure that on objects:

$$(45) \quad p_1 : (g_1, g_2) \Rightarrow (g_1, g_2)$$

$$(46) \quad p_2 : (g, g') \Rightarrow gg'$$

and on morphisms

$$(47) \quad p_1 : (h : (g_1, g_2) \rightarrow (hg_1g^{-1}, hg_2h^{-1})) \Rightarrow ((h, h) : (g_1, g_2) \rightarrow (hg_1h^{-1}, hg_2h^{-1}))$$

$$(48) \quad p_2 : (h : (g, g') \rightarrow (hgh^{-1}, hg'h^{-1})) \Rightarrow (h : gg' \rightarrow hgg'h^{-1})$$

The classes of connections on Y are of the form $[(g, g')]$ for $g, g' \in G$, and the class is an equivalence class modulo gauge transformations, conjugating by (h, h) . The classes for $S^1 \cup S^1$ are of the form $([g], [g'])$, since equivalence is by conjugation by $(h, h') \in G^2$. So connections which are gauge equivalent on $S^1 \cup S^1$ may be restrictions of inequivalent connections on Y .

Finally, suppose we have a functor $f : \mathcal{A}_0(S^1 \cup S^1)_G \rightarrow \mathbf{Vect}$: that is, a representation of $\mathcal{A}_0(S^1)_G \times \mathcal{A}_0(S^1)_G$, and transport it by $Z_G(Y) = (p_2)_* \circ (p_1)^*$. That is, we first pull back along p_1 from $S^1 \cup S^1$ to Y , then push forward along p_2 to S^1 .

Now, an irrep of $\mathcal{A}_0(S^1)_G \times \mathcal{A}_0(S^1)_G$ amounts to a pair of irreps of $\mathcal{A}_0(S^1)_G$. Each one amounts to a choice of isomorphism class $[g]$ in $G // G$, and a representation V of $\text{Aut}(g)$. Call these $([g], V)$ and $([g'], V')$. Then the pair $([g], [g'])$ is the image of any $[(g, g')]$ in $(G \times G) // G$ for some pair (g, g') representing $([g], [g'])$. This will then be pushed down by m to a representative of $\text{Aut}(g_1)$, where $g_1 = gg'$. There may be more than one $[(g, g')]$ for which this holds for a given $[g_1]$. In fact, following [36], we can then find that the image is:

$$(49) \quad (\pi_2)_* \circ \pi_1^*(g_1) \cong \bigoplus_{(g, g') \in ([g], [g']) | gg' = g_1} \mathbb{C}[\text{Aut}(g_1)] \otimes_{\mathbb{C}[\text{Aut}(g, g')]} (V \otimes V')$$

where the direct sum is over all non-equivalent (g, g') representing $([g], [g'])$ and satisfying $gg' = g_1$, and the action of G on each component is as we have described. On morphisms, we get the direct sum of the isomorphisms between these copies of \mathbb{C} . (This formula is a special case of the general form for Kan extension in an enriched category - see e.g. [14]).

We can say more about the example of Y by considering the situation for some particular groups.

Pants: Abelian Groups. It was established by Turaev [43] that a (2+1)-dimensional TQFT is determined by a modular tensor category (MTC) \mathbf{C} (i.e. monoidal category with a modular structure). In the framework we have been describing, $\mathbf{C} = Z_G(S^1)$ is the 2-vector space for the circle, with monoidal structure given by $\Lambda(Y) : \mathbf{C} \otimes \mathbf{C} \rightarrow \mathbf{C}$. This category has a finite set of generators (the number of generators is the “rank” of the MTC), and the monoidal structure is determined by the matrices representing $\Lambda(Y)$. In the case where G is abelian, this matrix is easy to describe: \mathbf{C} is of rank $|G|^2$, and the multiplication structure is quite simple.

The main point is that if G is abelian group, each $g \in G$ is in a separate conjugacy class. Moreover, the irreducible representations of G are 1-dimensional, and form \hat{G} , the character group of G , and in the abelian case $\hat{G} \cong G$.

For example, if $G = \mathbb{Z}_n$ is cyclic, then $\mathcal{A}_0(S^1)_G \cong \mathbb{Z}_n // \mathbb{Z}_n$ has objects the conjugacy classes of \mathbb{Z}_n . These are just the elements $\{0, \dots, n-1\}$, since G is abelian, and similarly the stabilizer groups are all $\text{Aut}(k) = \mathbb{Z}_n$. The irreducible representations for \mathbb{Z}_n are 1-dimensional, and form the character group $\hat{\mathbb{Z}}_n$. In character χ , the element $k \in \mathbb{Z}_n$ acts on \mathbb{C} by multiplication by χ^k for some n^{th} root of unity, of which there are n . So the generating objects of $Z_G(S^1)$ are labelled by $(m, n) \in \mathbb{Z}_n \times \hat{\mathbb{Z}}_n \cong \mathbb{Z}_n^2$.

Likewise, $\mathcal{A}_0(S^1 \cup S^1)_G = \mathbb{Z}_n^2 // \mathbb{Z}_n^2$, with objects labelled by (j, k) for $j, k \in \mathbb{Z}_n$. Again, representations are given by elements of the character group $\hat{\mathbb{Z}}_n^2$, and so $Z_G(S^1 \cup S^1)$ has generators in $\mathbb{Z}_n^2 \times \hat{\mathbb{Z}}_n^2 \cong \mathbb{Z}_n^4$.

For a general abelian group, we can use the above, in combination with the following straightforward fact:

Proposition 3. *If $G = G_1 \oplus \dots \oplus G_n$ is the direct sum of two abelian groups, then $\mathcal{A}_0(S^1)_G$ is:*

$$(50) \quad G // G \cong (G_1 // G_1) \times \dots \times (G_n // G_n)$$

Proof. The direct sum of abelian groups is a categorical product, and the group operation $+$ of G is by component-wise addition of elements (g_1, g_2) , so conjugation is also component-wise. So the objects of $G // G$ (conjugacy classes in G) correspond to pairs of conjugacy classes in G_1 and G_2 . The morphisms of $G // G$ are, again, stabilizers of these objects, which amount to all pairs of elements in G_1 and G_2 . By induction, this extends to p -fold direct sums. \square

Similarly, representations of a direct sum $G_1 \oplus G_2$ are tensor products of representations of G_1 and G_2 . These determine the bases of the 2-vector spaces $Z_G(S^1)$ and $Z_G((S^1)^2)$.

Theorem 3. *For G a finite abelian group, the matrix representation of $Z_G(Y)$ is given by the group multiplication for $G \times G$.*

Proof. More precisely, we begin by noting that $G \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_p}$ is a direct sum of cyclic groups.

So a representation supported on an object $(m, n) \in (G // G)^2$ pulls back to one supported on (m, n) in $(G \times G) // G$, and pushes forward to one supported on $m + n \in G // G$. An irreducible representation of $(G // G)^2$ amounts to a choice of object (m, n) and irreducible representation of $\text{Aut}(m, n) \cong G \times G$, which is just $\phi_m \otimes \phi_n$. An irreducible representation of $(G // G)^2$ amounts to a choice of object (m, n) and irreducible representation of $\text{Aut}(m, n) \cong G \times G$, which is just (ϕ_m, ϕ_n) . The 2-linear map Z_G takes this to $\phi_m \otimes \phi_n \cong \phi_{m+n}$.

Then the 2-linear map $Z_G(Y)$ comes from the span (44). The objects of $(G \times G)//G$ are again pairs $(m, n) \in G^2$, and $Aut(m, n) = G$ itself. On objects, the functor $(G \times G)//G \xrightarrow{\Delta} (G//G)^2$ is just the identity, and the map to $G//G$ is given by the group operation $+$.

Thus, the matrix representation for the 2-linear map $Z_{\mathbb{Z}_n}(Y)$ is an n^4 -by- n^2 matrix, which may be conveniently written as the tensor product of two copies of the "multiplication" matrix M_G for the group operation $+$. \square

In the case $G = \mathbb{Z}_2$, and using dimensions instead of vector spaces as components for convenience this is:

$$(51) \quad M_{\mathbb{Z}_2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

since this describes both multiplication (the part derived from the object map) and tensor product of representations (the part derived from morphisms).

This gives $Z_{\mathbb{Z}_2}(Y) = M_{\mathbb{Z}_2} \otimes M_{\mathbb{Z}_2}$, which is

$$(52) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, we have:

$$(53) \quad M_{\mathbb{Z}_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

giving $Z_{\mathbb{Z}_3}(Y) = M_{\mathbb{Z}_3} \otimes M_{\mathbb{Z}_3}$. By the above arguments about direct sums, if $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_p}$, the 2-linear maps for any abelian group can be obtained from such basic matrices by further tensor products.

Pants: Nonabelian Examples. The nonabelian case has two main features which do not appear in the abelian situation. First, the conjugacy classes of connections are less trivial, and not all stabilizer groups are all of G . Second, there will be cases where more than one object in the span contributes to the final result. We will see these properties in some explicit calculations. The matrix form of 2-linear maps $Z_G(Y)$ are in general rather large even for fairly small G (the MTC has large rank), so, we will find only certain interesting parts of these matrices.

In particular, we will look at parts of the matrix form of $Z_{S_3}(Y)$ and $Z_{S_4}(Y)$, which demonstrate both of the properties just mentioned. Since in general these matrices are quite large, we will simply find a few blocks.

Example 3. First, find $Z_{S_3}(S^1) = Rep(S_3//S_3)$. As usual, this is generated by irreducible objects labelled by $([g], \rho)$, where $[g]$ is a conjugacy class in $G = S_3$, and ρ is an irreducible representation of $Stab(g) \subset G = S_3$. The groupoid $S_3//S_3$ has the following groups and irreducibles: For $[t]$ and $[s]$, the stabilizer groups are abelian, so for \mathbb{Z}_2 we have the trivial and sign representations, and for $[s]$ we have the irreps of \mathbb{Z}_3 on \mathbb{C} as before. For S_3 , the three irreducible reps are labelled by three-block Young tableaux (see e.g. [41]), though these include the trivial representation $\mathbb{C} =$

$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$ and the sign representation $\sigma = \begin{array}{|c|} \hline \\ \hline \end{array}$. (The remaining irreducible representation, $\Gamma = \begin{array}{|c|c|} \hline & \\ \hline \end{array}$ is the 2-dimensional representation of S_3 given by the action of S_3 on the vertices of a triangle.)

Class $[g]$	Group $Stab(g)$	Representations
1 (identity)	S_3	$\mathbb{C}, \Gamma = \begin{smallmatrix} \square & \\ & \square \end{smallmatrix}, \sigma$
$[t]$ (transposition)	$\mathbb{Z}_2 = \{1, t\}$	\mathbb{C} and σ
$[s]$ (3-cycle)	$\mathbb{Z}_3 = \{1, s, s^2\}$	\mathbb{C}, ϕ, ϕ^2

TABLE 1. Generators of $Z_{S_3}(S^1)$

Then $Z_{S_3}(Y) : Z_{S_3}(S^1 \amalg S^1) \rightarrow Z_{S_3}(S^1)$ is a 2-linear map taking representations of $(S_3//S_3)^2$ to representations of $S_3//S_3$, which are as described above. An irreducible representation of $(S_3//S_3)^2$ is labelled by a pair $(([g], \rho), ([g'], \rho'))$ of irreps of $S_3//S_3$.

The functor $Z_{S_3}(Y)$ can then be described by a (64-by-8) matrix of vector spaces. The entries are given by Frobenius reciprocity, pulling representations back along m and Δ to representations of $\mathcal{A}_0(Y)_G = (S_3 \times S_3)//S_3$. So in particular, we get a sum over isomorphism classes in $\mathcal{A}_0(Y)_G$. These are: Notice that, since both

$[(g_1, g_2)]$	$Stab([(g_1, g_2)])$	$Im(m)$	$Im(\Delta)$
$[(1, 1)]$	S_3	$[1]$	$([1], [1])$
$[(1, t)]$	\mathbb{Z}_2	$[t]$	$([1], [t])$
$[(1, s)]$	\mathbb{Z}_3	$[s]$	$([1], [s])$
$[(t, 1)]$	\mathbb{Z}_2	$[t]$	$([t], [1])$
$[(t, t)]$	\mathbb{Z}_2	$[1]$	$([t], [t])$
$[(t, t')]$	1	$[s]$	$([t], [t])$
$[(t, s)]$	1	$[t]$	$([t], [s])$
$[(s, 1)]$	\mathbb{Z}_3	$[s]$	$([s], [1])$
$[(s, t)]$	1	$[t]$	$([s], [t])$
$[(s, s)]$	\mathbb{Z}_3	$[s]$	$([s], [s])$
$[(s, s^2)]$	\mathbb{Z}_3	$[1]$	$([s], [s])$

TABLE 2. Isomorphism Classes of $(S_3 \times S_3)//S_3$

elements of a pair (g_1, g_2) are conjugated by the same g in this quotient action, we can distinguish cycles and permutations, as in $([s], [s^2])$. Thus, there are two possible preimages of $([t], [t])$, depending on whether the two permutations t and t' are the same, and similarly for $([s], [s])$.

So then we have in matrix form $Z_{S_3}(Y)_{([g], \rho), ([g_1], \rho_1), ([g_2], \rho_2)}$ is given by a direct sum over the isomorphism classes in $(S_3 \times S_3)//S_3$ lying over $[g]$ by m and over $([g_1], [g_2])$ by Δ (that is, non-conjugate pairs with $g_2 g_1 = g$). The coefficients for particular representations show the multiplicity of ρ in the image of (ρ_1, ρ_2) . Unlike the abelian case, there are nontrivial coefficients from $([t], [t])$ to two different elements, $[s]$ and $[1]$.

For example, we will find the block of $Z_{S_3}(Y)$ corresponding to the objects $([t], [t])$ and $[1]$. There is a single object in $(S_3 \times S_3)//S_3$ lying over these objects, namely $[(t, t)]$. Restricting to these objects, we have the span of automorphism groups:

$$(54) \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \xleftarrow{\Delta} \mathbb{Z}_2 \xrightarrow{i} S_3$$

(The map $i : \mathbb{Z}_2 \rightarrow S_3$ is the injection homomorphism which takes the non-identity element of \mathbb{Z}_2 to $t \in S_3$.) Then we can calculate the block of $Z_{S_3}(Y)$ with indices given by the irreps of these groups, namely $\{(\mathbb{C}, \mathbb{C}), (\mathbb{C}, \sigma), (\sigma, \mathbb{C}), (\sigma, \sigma)\}$, and $\{\mathbb{C}, \Gamma, \sigma\}$, respectively.

We find these by pulling back each representation to \mathbb{Z}_2 along Δ or i , and using Schur's lemma. One can easily find the block to be (using integers to represent vector spaces):

$$(55) \quad Z_{S_3}(Y)_{1,([t],[t])} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

In the example above, we see that the matrix form of $Z_G(Y)$ need not be a multiplication matrix for a group, as it is for the abelian case, essentially because the image of an irreducible may not itself be irreducible. However this example is still special in that only one object in the middle groupoid $G \times G // G$ contributes to any given matrix entry. This is not true in general. The following (abbreviated) example illustrates the point in a more general framework than the above.

Example 4. If $G = S_4$, then $\mathcal{A}_0(S^1)_G = S_4 // S_4$ has isomorphism classes given by the conjugacy classes of permutations of 4 elements. These are classified by 4-box young diagrams, of which there are five:

$$(56) \quad \begin{array}{c} \square\square\square\square, \square\square\square, \square\square, \square\square, \square \\ \square\square\square, \square\square, \square, \square, \square \end{array}$$

In the same way, $\mathcal{A}_0(S^1 \cup S^1)_G$ has isomorphism classes given by pairs of such diagrams. Then as usual,

As in Example 3, we find a single block of $Z_{S_4}(Y)$, namely the block corresponding to $((\square\square\square, \square\square\square), \square\square\square)$. Here we are using the diagram which corresponds to the conjugacy class of a 3-cycle (i.e. a permutation of four elements with one fixed point). In the usual cycle notation for permutations, this object is $[(123)(4)]$.

The point now is that there are two classes in $(S_4 \times S_4) // S_4$ which project to $[(123)(4), (123)(4)]$ under Δ and $[(123)(4)]$ under the multiplication map m . That is, there are two conjugacy classes of pairs of 3-cycles whose product is a 3-cycle. Representatives of these two classes are: $a = ((123)(4), (123)(4))$, where the product is $m(a) = (132)(4)$; and $b = ((123)(4), (243)(1))$, where the product is $m(b) = (143)(2)$. It is straightforward to check these are the only cases.

Now, $Aut((123)(4)) \cong \mathbb{Z}_3$: precisely the powers of this 3-cycle stabilize it under conjugation (4 must be a fixed point of any $\pi \in Aut((123)(4))$, and a transposition would change the cycle). So, since a consists of two copies of this cycle, $Aut(a) \cong \mathbb{Z}_3$ also. On the other hand, $Aut(b) = \{Id\}$: no permutation stabilizes both permutations in the pair b .

Thus, to find the $((\square\square\square, \square\square\square), \square\square\square)$ block of $Z_{S_4}(Y)$, we take a direct sum over a and b of the restriction-induction functors. These come from two spans of automorphism groups from \mathbb{Z}_3^2 to \mathbb{Z}_3 . One can check that at a we have the span:

$$(57) \quad \mathbb{Z}_3^2 \xleftarrow{\Delta} \mathbb{Z}_3 \xrightarrow{id} \mathbb{Z}_3$$

The corresponding 2-linear map is just the multiplication map (53). On the other hand, at b we have the span:

$$(58) \quad \mathbb{Z}_3^2 \xleftarrow{i} \{Id\} \xrightarrow{i} \mathbb{Z}_3$$

In each case, the maps are the inclusions of the identity. Since the representations of \mathbb{Z}_3 (i.e. characters in $\hat{\mathbb{Z}}_3$) all pull back to the unique, trivial, representation of $\{Id\}$, Schur's lemma says the resulting matrix has \mathbb{C} in each component. So taking the direct sum over a and b , we find the block is:

$$(59) \quad \begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \end{pmatrix}$$

The other blocks are all found in a similar way (though we note that this is the only block for the case $G = S_4$ where more than one object appears in the direct sum.)

The final aspect of our weak 2-functor is its effect on 2-morphisms, and the next section gives examples of this.

4.3. Z_G on Cobordisms of Cobordisms. Now we consider the situation of a cobordism M between cobordisms. Applying the functor $\mathcal{A}_0(-)_G$, we get first a span of spans of groupoids of connections, as in (13). Then the 2-morphism $Z_G(M)$ is given by the unit and counit of the two adjunctions and the Nakayama isomorphism, as in equation (26).

In the case where objects in \mathbf{nCob}_2 are empty manifolds \emptyset , cobordisms between two empty sets are themselves manifolds (with empty boundary), and cobordisms between these have boundary, but no nontrivial corners. So we have just a cobordism from one manifold to another. It is reasonable to expect that in this case, the extended TQFT based on a group G should give results equivalent to those obtained from a TQFT based on the same group, suitably reinterpreted. This is indeed the case.

Example 5. Suppose we have two cobordisms S and S' from \emptyset to \emptyset , and a cobordism with (empty!) corners $M : S \rightarrow S'$. In fact, M should be thought of as a cobordism between manifolds, in a precisely analogous way that $Z_G(S)$ can be thought of as a TQFT giving a vector space for the manifold S .

In particular, we have

$$(60) \quad Z(S) \cong (- \otimes \mathbb{C}^k)$$

and

$$(61) \quad Z(S') \cong (- \otimes \mathbb{C}^{k'})$$

The k and k' are the number of isomorphism classes of connections on S and S' respectively. If we think of these as being vector spaces \mathbb{C}^k and $\mathbb{C}^{k'}$ assigned by a TQFT, then a cobordism should assign a linear map $T : \mathbb{C}^k \rightarrow \mathbb{C}^{k'}$. Indeed, such a linear map will give rise to a natural transformation from $Z(S)$ to $Z(S')$ by giving, for any objects $V \in \mathbf{Vect}$ on the left side of the diagram, the map $1_V \otimes T$ on the right side. Moreover, all such natural transformations arise this way.

A cobordism between cobordisms gives rise to a natural transformation:

$$(62) \quad \begin{array}{ccc} & (\pi_2)_* \circ \pi_1^* & \\ & \curvearrowright & \\ \mathbf{Vect} & \begin{array}{c} \downarrow \\ Z(M) \\ \downarrow \end{array} & \mathbf{Vect} \\ & \curvearrowleft & \\ & (\pi'_2)_* \circ (\pi'_1)^* & \end{array}$$

As discussed in [36], this reduces to the groupoidification functor

$$(63) \quad D : \text{Span}(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$$

and our construction just yields a TQFT. That is, each 2-linear map can then be described as a 1-by-1 matrix of vector spaces (that is, a vector space), and the natural transformations are just described in this one component by a single linear map.

Now we look, in the 3D case, at a more general 2-morphism M in \mathbf{nCob}_2 and find $Z_G(M)$ for a general G .

Example 6. In figure 2 we show an example, which can be construed as taking a pair of pants Y to its reversed version.

FIGURE 2. A 2-Morphism in $\mathbf{3Cob}_2$

Of course, since both source and target change in this process, this is actually drawn as a 2-morphism in the double bicategory, which we can represent as a diagram like the following in $\mathbf{ManCorn}$ (noting that all the arrows may be taken as collarable inclusions):

$$(64) \quad \begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A \amalg D) & \xleftarrow{i'_A \otimes i_D} & S^1 \amalg S^1 \\ i_1 \downarrow & & \downarrow \iota_1 & & \downarrow i_2 \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ i_2 \uparrow & & \uparrow \iota_2 & & \uparrow i_1 \\ S^1 \amalg S^1 & \xrightarrow{i_2} & Y & \xleftarrow{i_1} & S^1 \end{array}$$

Here, A is the annulus and D the disk in the top horizontal cobordism, the Y are instances of the pair of pants, and M is the whole 3-dimensional manifold with

corners. The leftmost vertical cospan is the inner cobordism, and the rightmost is the outer. The maps come from the obvious inclusions.

To compute $Z_G(M)$, we must convert M to a 2-morphism in the *bicategory* $\mathbf{3Cob}_2$. However, for the moment, we can describe the effect of Z_G on this square. First, applying $\mathcal{A}_0(-)_G$ to (64), we get a diagram equivalent to the following double cospan of groupoids:

$$(65) \quad \begin{array}{ccccc} G//G & \xleftarrow{id} & (G//G) \otimes 1 & \xrightarrow{id \otimes !} & (G//G)^2 \\ \uparrow m & & \uparrow (m,! & & \uparrow \Delta \\ G^2//G & \xleftarrow{id} & G^2//G & \xrightarrow{(m,1)} & G^2//G \\ \Delta \downarrow & & id \downarrow & & \downarrow m \\ (G//G)^2 & \xleftarrow{\Delta} & G^2//G & \xrightarrow{m} & G//G \end{array}$$

The labelling here uses a presentation of $\mathcal{A}_0(M)_G$ where connections are labelled by the holonomies around the two circles at the bottom, which is sufficient to describe them since this M is homotopic to Y . In particular, notice that while M , and each copy of Y , all have equivalent groupoids of flat connections, the maps which join these are nontrivial. (In particular, the map $(m, 1)$ from M to the outside Y reflects the fact that any flat connection on M will restrict to have trivial holonomy around the right-hand top circle, since this is contractible in M .)

Now, construing this as a 2-morphism in the *bicategory* $\mathbf{3Cob}_2$, we can compose around the corners of the square, to get two spans from the upper left to lower right (from $\mathcal{A}_0(S^1)_G \simeq G//G$ to itself). Then M becomes a cospan of cospan maps, forming a 2-morphism between the two cobordisms:

$$(66) \quad Y \circ (A \amalg D) \rightarrow M \leftarrow Y \circ Y^\dagger$$

as illustrated in Figure 3. Here, Y^\dagger is the adjoint of Y as a cobordism, namely Y with direction reversed. This step of the construction of Z_G is slightly awkward since \mathbf{nCob}_2 is most generally a double bicategory, that is, a weak cubical 2-category, and $\mathbf{2Vect}$ is most naturally a bicategory, that is, a weak globular 2-category. As mentioned in Section 2.1, this bicategory is equivalent to the more commonly used form of \mathbf{nCob}_2 . We have chosen this method of reconciling them rather than the alternate approach of making a cubical version of $\mathbf{2Vect}$, simply because cubical n -categories are less familiar.

FIGURE 3. The Same 2-Morphism, In A Bicategory

The left-hand composite S just amounts to the identity cobordism for S^1 . So in particular, we can also regard this calculation as giving a cobordism from the identity to the pair of pants. Applying $\mathcal{A}_0(-)_G$ gives a span of connection groupoids, all $G//G$, with two identity maps.

The right-hand cobordism is the composite $S' = Y^\dagger \circ Y$. The corresponding span of groupoids can be found by applying $\mathcal{A}_0(-)_G$ directly (as we will do here), or by taking the weak pullback of the spans for the pants cobordisms. Applying $\mathcal{A}_0((-)_G)$, we note that the cobordism is a 1-torus with two punctures. Its fundamental groupoid is equivalent to the fundamental group $\pi_1(S') = F(x_1, x_2, x_3)$, the free group on three generators. Here, we take x_1 and x_2 to be homotopy class of paths around the two “legs”, and x_3 to be the class of paths around the hole in the torus. A functor from this into G is determined by the images of the generators, with gauge transformations acting by conjugation at one point. Thus, $\mathcal{A}_0(S')_G \simeq (G \times G \times G)//G$

So finally we have the span of span maps:

$$(67) \quad \begin{array}{ccccc} & & G//G & & \\ & id \swarrow & \uparrow m & \searrow id & \\ G//G & \xleftarrow{m} & (G^2)//G & \xrightarrow{m} & G//G \\ & \swarrow s & \downarrow i_3 & \searrow t & \\ & & (G^3)//G & & \end{array}$$

where the maps are given by:

$$(68) \quad i_3 : (g_1, g_2) \mapsto (g_1, g_2, 1)$$

and

$$(69) \quad s : (g_1, g_2, g_3) \mapsto g_2^{-1} g_1$$

and

$$(70) \quad t : (g_1, g_2, g_3) \mapsto g_2^{-1} g_3 g_1 g_3^{-1}$$

Composing the 2-linear maps given in Sections 4.2 and 4.2 for the pair of pants Y and its dual (which is the dual 2-linear map) gives $Z_G(S')$ directly. Following [36], the natural transformation $Z_G(M) : Z_G(S) \rightarrow Z_G(S')$ is:

$$(71) \quad Z_G(M) = \epsilon_{L, i_3} \circ \eta_{R, m}$$

For a connection a on the source and representation F of $G//G$ (the central groupoid of the top span), these maps do the following to a vector v in the representation space $F(a)$:

$$(72) \quad \eta_{R, m}(F)(a) : v \mapsto \bigoplus_{[A] | m(y) \cong a} \frac{1}{\#Aut(y)} \sum_{h \in Aut(a)} h^{-1} \otimes h(v)$$

and

$$(73) \quad \epsilon_{L, i_3}(F')(a') : \bigoplus_{[A] | i_3(A) \cong a'} h_A \otimes v \mapsto \sum_{[A] | i_3(A) \cong x} i_3(h_A)v$$

(Objects A are connections on M). So the composite gives:

(74)

$$Z_G(M)(F)(a)(v) = \bigoplus_{a' | i_3(A) \cong a', m(A) \cong a} \frac{1}{\# \text{Aut}(A)} \sum_{[A] | i_3(A) \cong a'} \sum_{h \in \text{Aut}(A)} i_3(h^{-1})h(v)$$

In particular, a connection $a \in G//G$ is given by one group element $g \in G$, so this amounts to:

$$(75) \quad Z_G(M)(F)(g)(v) = \bigoplus_{g' \in G} \frac{1}{\# \text{Aut}(g', g'g^{-1})} \sum_{h \in \text{Aut}(g', g'g^{-1})} v$$

Note that in this case, the only objects of $G^3//G$ with nontrivial contribution are those of the form $(g', g'g^{-1}, 1)$, and i_3 takes a gauge transformation represented by h to one represented by the same h . Thus, $i_3(h^{-1})h(v) = v$.

A seemingly simpler example demonstrates that one cannot necessarily reduce the fundamental groupoid to a fundamental group.

Example 7. The *Dehn twist* is a fundamental generator of the 3D cobordism 2-category. It is the 3D cobordism with corners which takes a cylinder $S^1 \times I$ to a “twisted” cylinder (this unfortunate terminology for a purely topological “twist” should not be confused with the cocycle twisting discussed in section 5). Intuitively, the twisted cylinder is a version of $S^1 \times [0, 1]$ which is parameterized differently, by the map $i_s : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$, with:

$$(76) \quad i_s : (\phi, s) \mapsto (\phi \cdot e^{2\pi i s}, s)$$

That is, the Dehn twist is a boundary-preserving diffeomorphism of the cylinder with itself: this reparametrization fixes the boundaries at $s = 0$ and $s = 1$. The need for the Dehn twist reflects the fact that the 3-dimensional rotation group is not simply connected, so there is no continuous deformation from this map to the identity. On the other hand, since the fundamental group of $SO(3)$ is \mathbb{Z}_2 , two copies of the Dehn twist, or equivalently, a reparametrization by $\phi \mapsto \phi \cdot e^{4\pi i s}$, can be continuously deformed to the identity. Thus, the identity 2-cell in \mathbf{nCob}_2 (which is taken only up to diffeomorphism) connects one to the other.

The Dehn twist, however, is not the same (up to diffeomorphism) as the identity. Instead, it is represented as the mapping cylinder with each parameterization included as one horizontal face, and identities on the vertical faces:

$$(77) \quad \begin{array}{ccccc} S^1 & \xrightarrow{s=0} & S^1 \times [0, 1] & \xleftarrow{s=1} & S^1 \\ t=0 \downarrow & & t=0 \downarrow & & t=0 \downarrow \\ S^1 \times [0, 1] & \xrightarrow{s=0} & S^1 \times [0, 1]^2 & \xleftarrow{s=1} & S^1 \times [0, 1] \\ t=1 \uparrow & & i_s \times id \uparrow & & s=1 \uparrow \\ S^1 & \xrightarrow{s=0} & S^1 \times [0, 1] & \xleftarrow{t=1} & S^1 \end{array}$$

Taking groupoids of connections, the twist is difficult to represent using a skeletal form of the fundamental groupoids, as in the previous example. Of course, any groupoid is equivalent to a skeletal one, but for clarity, it is easier to pass through a representative of the fundamental groupoid in which there is a base-point in

each circle (i.e. boundary component for the cylinders). Then the groupoid of connections is represented as the category of functors into G .

These are characterized by the holonomies along some generating paths, with gauge transformations acting by natural transformations - i.e. by “conjugation” at the base-points. There are various possible choices, which give equivalent groupoids with different notation. One choice for $S^1 \times [0, 1]^2$ is to take as base-points $1 \in S^1$ at each corner, and as generators the following paths:

$$(78) \quad \{(\phi, 0, 0) | \phi \in S^1\}$$

$$(79) \quad \{(\phi, \phi \cdot e^{4\pi i s}, 1) | \phi \in S^1, s \in [0, 1]\}$$

$$(80) \quad \{(1, s, 0) | s \in [0, 1]\}$$

$$(81) \quad \{(1, 0, t) | t \in [0, 1]\}$$

Corresponding presentations are used for the groupoids associated to the four copies of $S^1 \times [0, 1]$. Then we will present the connection groupoids in terms of holonomies along these paths (and the analogous paths in the faces). $\mathcal{A}_0(S^1)_G = G//G$, $\mathcal{A}_0(S^1 \times [0, 1])_G = G^2//G^2$, and $\mathcal{A}_0(S^1 \times [0, 1]^2)_G = G^4//G^4$, where the actions are as described above. These groupoids are all equivalent to $G//G$, and in particular the representation categories will also be equivalent. The information which captures the distinctive properties of the Dehn twist, however, is in the maps between these groupoids. In the presentation given above, most of these maps are just projections onto individual components or else, for instance, the conjugation:

$$(82) \quad ((g_1, g_2, g_3, g_4), (h_{00}, h_{01}, h_{10}, h_{11})) \mapsto ((g_3 g_1 g_3^{-1}, g_2 g_3 g_1^{-1} g_4^{-1}, (h_{10}, h_{11}))$$

corresponding to pulling back holonomies along the inclusion $s = 1$. The first component finds the holonomy around the initial circle on the $s = 1$ cylinder (by conjugating with the path between base-points). The second computes the holonomy along the path $\{(1, 1, t) | t \in [0, 1]\}$ via a homotopic path. The h_{ij} label a morphism by the stages of a natural transformation at the base-point on the $(s, t) = (i, j)$ circle.

Pulling back along the inclusion i_s is just:

$$(83) \quad ((g_1, g_2, g_3, g_4), (h_{00}, h_{01}, h_{10}, h_{11})) \mapsto ((g_3 g_1 g_3^{-1}, g_2), (h_{01}, h_{11}))$$

The remainder of the computation is of substantially similar sort to the previous example.

5. TWISTING AND 2-LINEARIZATION

We now would like to see how our categorified quantization process $\Lambda : \text{Span}(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$ generalizes to a twisted 2-linearization, $\Lambda^{U(1)}$. Then our main result generalizes to a claim that the twisted DW model is a composite of $\Lambda^{U(1)}$ and a “classical field theory” valued in groupoids equipped with such cocycles. These classical field theories are then classified by choices (G, ω) , where G is a finite group and $\omega \in H_{grp}^3(G, U(1))$ is a cohomology class. The “quantization functor” $\Lambda^{U(1)}$ will be the same for all such ETQFTs.

5.1. Cocycle Twisting as HQFT. The above remarks imply that the action functional associated to ω will be part of the classical theory, and $\Lambda^{U(1)}$ will be the same 2-functor for all choices of ω . As we shall see, the most natural way to do this uses a generalization of $\text{Span}(\mathbf{Gpd})$, which we will call $\text{Span}(\mathbf{Gpd})^{U(1)}$, in which groupoids may come equipped with some $U(1)$ -cocycle information, which

$\Lambda^{U(1)}$ will respect. This contains an isomorphic copy of $\text{Span}(\mathbf{Gpd})$ as the sub-2-category where this extra data is trivial - that is, all cocycles take the constant value $1 \in U(1)$ - on which $\Lambda^{U(1)}$ restricts to Λ .

This cohomological aspect of the construction of TQFT from Lie groups has been developed in detail by Freed, Hopkins, Lurie and Teleman [16]. We remark that this framework focuses especially on classifying TQFTs in dimension n by means of an element of the n^{th} group cohomology of G , so since Λ and $\Lambda^{U(1)}$ are specifically 2-categorical our construction gives a version of this theory which extends only to codimension-2. The reconstruction of the DW model when $n = 3$ is of special interest since 1D manifolds are sufficiently simple to be an interesting stopping point.

Now, recall that a group cohomology element is an element of the ordinary cohomology of the classifying space:

$$(84) \quad [\omega] \in H^3(BG, U(1))$$

Its role is best understood in terms of the fact that G -connections on a manifold M correspond to homotopy classes of maps $A : M \rightarrow BG$ into the classifying space of G . This applies to both manifolds and cobordisms, so we may understand the role of $[\omega]$ in the context of the Homotopy Quantum Field Theory of Turaev [44]. An HQFT is rather like a TQFT, but the source category consists not just manifolds and cobordisms, but of manifolds and cobordisms *equipped* with maps into a target space X . In the case where $X = BG$, this says that an HQFT is an assignment of vector spaces and linear maps to manifolds and cobordisms equipped with a G -bundle with connection.

Such HQFTs are classified precisely by cohomology classes on X (or BG in this case), or equivalently in terms of gerbes on X . This sort of construction, in a somewhat more specialized case, has been described by Picken [38], using the language of TQFT with a more general source category to describe what is in fact an HQFT. There, such a construction is described explicitly in terms of gluing rules for manifolds with boundary, rather than categorically in terms of composition of cobordisms. Moreover, these are labelled with specific collections of neighborhoods, and the abstract cohomology class $[\omega]$ becomes concrete transition functions for a gerbe relative to these neighborhoods. Despite these differences, the formulation given there is helpful in understanding the composition in the new 2-category $\text{Span}(\mathbf{Gpd})^{U(1)}$ introduced in Section 5.2.

Now, in the untwisted situation, the classical field theory $\mathcal{A}_0(-)_G$ assigns a groupoid $\mathcal{A}_0(S)_G$ of all connections, which are given by homotopy classes of maps into BG (connections) for each manifold S . The exact correspondence is that the mapping space $\text{Maps}(S, BG)$ is the classifying space for this groupoid. In the twisted case, $\mathcal{A}_0(-)_G^\omega$ will produce the same groupoid, but associate cocycle information derived from ω , as described below following the description in [45]. Then, where Λ simply performs a sum (or direct sum) over all the objects of this groupoid in a span, the twisted form $\Lambda^{U(1)}$ will modify these sums using this cocycle information.

Now we describe the category of groupoids with cocycle which we need to make this work.

5.2. The 2-category $\text{Span}(\mathbf{Gpd})^{U(1)}$. In a previous work [34], the author described a monoidal category of groupoids with phases valued in $U(1)$, a special

case of groupoids with labels valued in a monoid M . The motivation there was to allow for a more physically realistic model of the quantum harmonic oscillator in a category of groupoids and spans. The $U(1)$ -phases were needed to get interesting time evolution operators. This involved spans of groupoids equipped with phases derived from a “number-operator”, which plays the role of a *Hamiltonian* for that system. Now we want to describe a variation on this, thinking of phases in the *Lagrangian* sense, as (exponentiated) actions rather than energies, but an analogous structure is required.

We wish to expand our factorization $\Lambda \circ \mathcal{A}_0(-)_G = Z_G(-)$ to include the twisted case, where a nontrivial Lagrangian i . So there should be a factorization through a 2-category of which contains $\text{Span}(\mathbf{Gpd})$. Our 2-category $\text{Span}(\mathbf{Gpd})^{U(1)}$ will look like $\text{Span}(\mathbf{Gpd})$, except that groupoids come equipped with some extra data.

The simplest such data, most obviously related to Lagrangians in the standard physical sense, is the assignment of an (exponentiated) action in $U(1)$ to a history for a system. That is, the $U(1)$ -element is assigned to an object of the groupoid in the middle of a 2-morphism. This function ought to be an invariant of isomorphism classes of objects (physically indistinguishable histories get the same action). When we apply Λ to a 2-morphism, we sum over such “histories”. In [34] it is explained how groupoidification can be extended to replace groupoid cardinality with a weighted sum:

$$(85) \quad |(\mathcal{G}, f)| = \sum_{[x] \in \underline{\mathcal{G}}} \frac{f(x)}{|\text{Aut}(x)|}$$

This is naturally found in $\mathbb{R}^+ \otimes U(1)$, which we map into \mathbb{C} (identifying all elements $0, \phi$) with $0 \in \mathbb{C}$).

This allowed the construction of a full complex Hilbert space in [34]. Now, the role of cardinality in groupoidification arises from the Nakayama isomorphism between the left and right adjoints of the restriction functors. In $\text{Hom}(\mathbf{1}, \mathbf{1})$, as discussed in [36], this isomorphism simply becomes a numerical factor, the groupoid cardinality. Thus, as might be expected, our $\Lambda^{U(1)}$ will incorporate the $U(1)$ -valued topological action into a twisting of the Nakayama isomorphism, at the 2-morphism level.

In building $\text{Span}(\mathbf{Gpd})^{U(1)}$ as a monoidal 2-category, it is not sufficient simply to take spans in $U(1) - \mathbf{Gpd}$: that is, groupoids with $U(1)$ -functions on them. Instead, we need a different structure to describe the appropriate “categorification of the action functional”, which will reproduce the twisted DW model. The key point is that $U(1)$ phases on objects can be understood as 0-cocycles in groupoid cohomology.

With our overall aim in mind, we will define a 2-category in which this classical process takes values:

Definition 9. *The monoidal 2-category $\text{Span}(\mathbf{Gpd})^{U(1)}$ has:*

- **Objects:** *groupoids A equipped with 2-cocycle $\theta \in Z^2(A, U(1))$*
- **1-Morphisms:** *a morphism from (A, θ_A) to (B, θ_B) is a span of groupoids $A \xleftarrow{s} X \xrightarrow{t} B$, equipped with 1-cocycle $\alpha \in Z^1(X, U(1))$*
- **2-morphisms:** *a 2-morphism from (X, α, s, t) to (X', α', s', t') in $\text{Hom}((A, \theta_A), (B, \theta_B))$ is a class of spans of span maps $X \leftarrow Y \rightarrow X'$ equipped with 0-cocycle $\beta \in Z^0(Y, U(1))$, with equivalence taken up to β -preserving isomorphism of Y*

Subject to the conditions:

- In any 1-morphism

$$(86) \quad (X, \alpha, s, t) : (A, \theta_A) \rightarrow (B, \theta_B)$$

the cocycles satisfy

$$(87) \quad (s^* \theta_A) = (t^* \theta_B)$$

In particular, $[s^* \theta_A] = [t^* \theta_B]$

- In any 2-morphism

$$(88) \quad (Y, \beta, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \Rightarrow (X_2, \alpha_2, s_2, t_2)$$

the cocycles satisfy

$$(89) \quad (\sigma^* \alpha_1)(\tau^* \alpha_2)^{-1} = 1$$

(In particular, $[\sigma^* \alpha_1] = [\tau^* \alpha_2]$, but moreover, since a 0-cocycle on a groupoid is an invariant function, $\delta\beta = 0$ and the cocycles themselves are equal.)

The structures making $\text{Span}(\mathbf{Gpd})^{U(1)}$ a monoidal 2-category are:

- Composition of 1-morphisms

$$(90) \quad (X_1, \alpha_1, s_1, t_1) : (A, \theta_A) \rightarrow (B, \theta_B)$$

and

$$(91) \quad (X_2, \alpha_2, s_2, t_2) : (B, \theta_B) \rightarrow (C, \theta_C)$$

at the object (B, θ_B) gives the same span of groupoids as in $\text{Span}(\mathbf{Gpd})$, and assigns the pullback object the cocycle

$$(92) \quad \alpha_1 \cdot \alpha_2 \cdot \theta_B$$

(explained below)

- Vertical composition of 2-morphisms:

$$(93) \quad (Y, \beta, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \rightarrow (X_2, \alpha_2, s_2, t_2)$$

and

$$(94) \quad (Y', \beta', \sigma', \tau') : (X_2, \alpha_2, s_2, t_2) \rightarrow (X_3, \alpha_3, s_3, t_3)$$

at (X_2, α_2) gives the same groupoids as in $\text{Span}(\mathbf{Gpd})$, with the cocycle given by

$$(95) \quad \beta \cdot \beta' \cdot \alpha_2$$

- Horizontal composition of 2-morphisms:

$$(96) \quad (Y, \beta, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \rightarrow (X_2, \alpha_2, s_2, t_2)$$

in $\text{Hom}((A, \theta_A), (B, \theta_B))$, and

$$(97) \quad (Y', \beta', \sigma', \tau') : (X'_1, \alpha'_1, s'_1, t'_1) \rightarrow (X'_2, \alpha'_2, s'_2, t'_2)$$

in $\text{Hom}((B, \theta_B), (C, \theta_C))$ at (B, θ_B) gives the same groupoids as in $\text{Span}(\mathbf{Gpd})$, with the cocycle given by

$$(98) \quad \beta \cdot \beta' \cdot \theta_B$$

- The monoidal structure is given by

$$(99) \quad (A, \theta_A) \otimes (B, \theta_B) = (A \times B, \theta_A \cdot \theta_B)$$

We explicitly define the cocycle $\alpha_1 \cdot \alpha_2 \cdot \theta_B$ in (92) as follows. First recall that spans of groupoids are composed by taking the weak pullback of t_1 and s_2 , which is the iso-comma groupoid $t_1 \downarrow s_2$. Its objects are triples (x_1, f, x_2) where $f : t_1(x_1) \rightarrow s_2(x_2) \in B$, and its morphisms are pairs $(g_1, g_2) \in X_1 \times X_2$, forming commuting squares in B :

$$(100) \quad \begin{array}{ccc} s_1(x_1) & \xrightarrow{f} & t_2(x_2) \\ s_1(g_1) \downarrow & & \downarrow t_2(g_2) \\ s_1(x'_1) & \xrightarrow{f'} & t_2(x'_2) \end{array}$$

For 1-morphisms $(X, \alpha, s, t) : (A, \theta_A) \rightarrow (B, \theta_B)$, the defining property of the 1-cocycle α is that it determines a functor $\alpha : X \rightarrow U(1)$, where $U(1)$ is understood as a groupoid with one object. On the other hand, the 2-cocycle θ_B is a map from pairs of morphisms $(f, f') \in B$ to $U(1)$ satisfying the 2-cocycle property, which ensures that “twisting” multiplication by θ_B remains associative if it was so originally.

Then the functor $\alpha_1 \cdot \alpha_2 \cdot \theta_B : (t_1 \downarrow s_2) \rightarrow U(1)$ assigns the 1-morphism (100) the value:

$$(101) \quad \alpha_1(g_1) \cdot \alpha_2(g_2) \cdot \theta_B(f, f')$$

(This is the meaning of “twisting multiplication by θ_B ” above).

The horizontal composition rule for 2-morphisms is similar, in that we compose by weak pullback over (B, θ_B) for both 1-morphisms and 2-morphisms. All the twistings by θ_B are compatible under the maps of the 2-morphism’s inner span.

The vertical composition rule for 2-morphisms are likewise similar, except that the 0-cocycle is assigned to objects of $Y' \circ Y$. These are again of the form (y', f, y) where $f : \sigma'(y') \rightarrow \tau(y) \in X_2$. So the composite cocycle assigns this object the value $\beta'(y') \cdot \alpha_2(f) \cdot \beta(y)$.

This composition rule may be surprising at first sight, in that one might naively expect the cocycle on $Y' \circ Y$ to be $\beta' \cdot \beta$, or on $X_2 \circ X_1$ to be $\alpha_2 \cdot \alpha_1$, so that a 0-cocycle is a product only of 0-cocycles, and so on. It is clear, though, that objects in the $\tau \downarrow \sigma'$ contain a morphism from X_2 , or that $t_1 \downarrow s_2$ contains two morphisms from B , so this rule is clearly a consequence of the use of the *weak* pullback to compose spans of groupoids. We will describe this in terms of transition functions in the case of groupoids of connections in Section 5.3. For the moment, the intuitive idea is that in the composite of two spans of groupoids, objects in the Y or the X_i need not match exactly as in a fibered product, and some “twisting” is needed to align them correctly. As we will see, this is necessary to ensure that the composition rules indeed determine cocycles on the groupoids found by such weak pullback.

Then the main point is that:

Lemma 1. $\text{Span}(\mathbf{Gpd})^{U(1)}$ is a symmetric monoidal 2-category.

Proof. We have already noted that $\text{Span}(\mathbf{Gpd})$ is a symmetric monoidal 2-category (the symmetry is due to the fact that the monoidal product comes from the categorical product in \mathbf{Gpd}). Details of this proof are found in [36]. It remains to check that the properties still hold when cocycle data is added.

First, $\text{Span}(\mathbf{Gpd})^{U(1)}$ is closed under composition of 1-morphisms if (92) determines a 1-cocycle, namely $\alpha_1 \cdot \alpha_2 \cdot \theta_B$ is a functor from $(t_1 \downarrow s_2)$ into $U(1)$. Since

α_1 and α_2 are functors, this follows precisely from the fact that θ_B is a 2-cocycle, so that the twisted multiplication remains associative. In the same way, (95) determines an invariant function on objects of $(\tau \downarrow \sigma')$ (that is, a 0-cocycle), since α_2 is a 1-cocycle, so its coboundary is trivial. It remains to check that it satisfies the axioms of a 2-category. Composition in $\text{Span}(\mathbf{Gpd})$ has an associator isomorphism which is given by the universal property of the weak pullback. This isomorphism is simply a re-bracketing of the same elements, hence it preserves the cocycles, and induces an associator for composition of 1-morphisms in $\text{Span}(\mathbf{Gpd})^{U(1)}$. A similar argument shows that composition of 2-morphisms is associative.

The units of $\text{Span}(\mathbf{Gpd})^{U(1)}$ are just the usual units in $\text{Span}(\mathbf{Gpd})$ with trivial cocycles (i.e. constant of value 1).

The fact that (99) defines a monoidal product follows from the fact that the Cartesian product \times on \mathbf{Gpd} determines a monoidal product in $\text{Span}(\mathbf{Gpd})$, and multiplication is a monoid operation for $U(1)$. The fact that this monoidal product is symmetric follows from the symmetry isomorphism from the universal property of \times and commutativity of multiplication in $U(1)$. The monoidal unit is $(\mathbf{1}, 1)$, as can easily be verified. \square

An obvious but important fact is:

Corollary 2. *The symmetric monoidal 2-category $\text{Span}(\mathbf{Gpd})^{U(1)}$ contains $\text{Span}(\mathbf{Gpd})$ as a sub-symmetric monoidal 2-category.*

Proof. There is a fully faithful symmetric monoidal 2-functor embedding any object, morphism, or 2-morphism of $\text{Span}(\mathbf{Gpd})$ into $\text{Span}(\mathbf{Gpd})^{U(1)}$ taking any groupoid to the same groupoid equipped with the trivial cocycle which has the constant value 1, and leaving all maps in spans unchanged. This is clearly a functor, since all operations in $\text{Span}(\mathbf{Gpd})^{U(1)}$ are just the same as those in $\text{Span}(\mathbf{Gpd})$ when cocycles are disregarded. The image of this embedding is a sub-category since it contains all identities and the monoidal unit, and is closed under the composition and monoidal operations of $\text{Span}(\mathbf{Gpd})^{U(1)}$. \square

Knowing that $\text{Span}(\mathbf{Gpd})^{U(1)}$ is a symmetric monoidal 2-category, we want to construct the two symmetric monoidal 2-functors $\mathcal{A}_0(-)_G^\omega$ (given a fixed 3-cocycle $\omega \in Z^3(BG, U(1))$), and $\Lambda^{U(1)}$. We address these next.

5.3. The Classical Field Theory. Our construction of the ETQFT Z_G^ω corresponding to the twisted DW model will use a generalization of the factorization $Z_G(-) = \Lambda \circ \mathcal{A}_0(-)_G$. The quantization functor $\Lambda^{U(1)} : \text{Span}(\mathbf{Gpd})^{U(1)}$ will always be the same, and the cocycle ω will modify only the classical field theory, by defining the cocycle data in $\text{Span}(\mathbf{Gpd})^{U(1)}$.

We will begin by describing the classical field theory component.

The "topological action" for the twisted DW theory comes from a $U(1)$ -valued class in group cohomology:

$$(102) \quad [\omega] \in H_{gp}^3(G, U(1))$$

which we can take as represented by some particular cocycle:

$$(103) \quad \omega \in Z_{gp}^3(G, U(1))$$

Now, group cohomology is just the usual third cohomology of the classifying space of G , so in fact this says:

$$(104) \quad \omega \in Z^3(BG, U(1))$$

This is a function which, given a 3-cycle in the space BG , defines a number in $U(1)$, satisfying the cocycle condition. It is usual to think of the classifying space defined simplicially, and therefore to consider what ω does to 3-simplices.

Now, the classical part of the DW construction with associated 3-cocycle ω gives cocycles of different degree associated to the groupoids of connections for manifolds of different dimension. In general, a k -dimensional cobordism will produce a groupoid (as object, or part of a span) which has a $(3 - k)$ -cocycle associated to it. This data arise from the “transgression” of the original cocycle, an algebraic structure explained very nicely by Willerton [45]. We briefly summarize it here.

First, this concept involves the view of a (flat) connection on a k -dimensional manifold M can be understood as a (homotopy class of) map(s) into the classifying space, $f : M \rightarrow BG$. Recall that an indirect definition of BG is precisely this fact. The classifying space functor B is right adjoint to the fundamental-groupoid functor Π_1 , so that in particular $Hom(\Pi_1(M), G) \cong Hom(M, BG)$, where the second term consists of homotopy classes of maps of spaces. The most important feature of BG is that its fundamental group is G and all other homotopy groups are trivial. More concrete constructions of BG for particular G depend on exactly which category of spaces BG is considered to lie in.

One standard choice, used in the “bar construction” is that BG is a simplicial set. It can be constructed by taking a single base-point (if G is a group, or one base-point for each object if G is a groupoid), adding edges for each element of G , and then adjoining higher-dimensional cells as necessary to make sure there are no higher homotopy groups. For instance, one would add: a triangular face adjoining edges f , g and fg to make this loop contractible; a tetrahedron between four such triangles expressing each associativity relation; and so on. It will be convenient to take BG to be presented as such a simplicial complex. A 3-cocycle ω on BG then gives a value in $U(1)$ to each 3-simplex Δ_3 in BG . (A smooth realization of BG will treat this as an integral of some 3-form over the 3-chain Δ_3 . In general, when describing integration in the group $U(1)$, we will treat it as the additive group \mathbb{R}/\mathbb{Z}).

Now, a connection is given by point in the space $Maps(M, BG)$. This space in turn is a simplicial complex, and in fact is the classifying space of the groupoid of flat connections on M (each isomorphism class of objects corresponds to a connected component of this space given by one of the base-points). Since a point in $Maps(M, BG)$ is a function, there is the evaluation map:

$$(105) \quad ev : M \times Maps(M, BG) \rightarrow BG$$

The image of $f : M \rightarrow BG$, or rather of $M \times f$, under ev is then a k -chain (perhaps degenerate) in BG , namely the image $f(M)$. If we take a $(3 - k)$ -simplex Δ_{3-k} in $Maps(M, BG)$, then these images in BG form a 3-dimensional subspace which looks like $M \times \Delta_{3-k}$. This can be decomposed into individual simplices in BG .

But then, this means we have a $(3 - k)$ -cocycle on $Maps(M, BG)$, the “transgression” of ω to $Maps(M, BG)$, which is denoted:

$$(106) \quad \tau_M(\omega) \in H^{3-k}(Maps(M, BG), U(1))$$

It is given by integrating ω :

$$(107) \quad \tau_M(\omega) = \int_M ev^*(\omega)$$

It can be integrated over Δ_{3-k} to get an element of $U(1)$. Our classical field theory will assign the cocycle $\tau_M(\omega)$ to each groupoid $\mathcal{A}_0(M)_G$.

So for 2-morphisms in $\mathbf{3Cob}_2$, which are 3-dimensional cobordisms M ("space-times with boundary"), this just amounts to an action functional: a $U(1)$ -valued function on connections. In the untwisted case, we have the constant function $\omega \cong 1$, and thus $\tau_M(\omega) \cong 1$ also. But for objects (1-dimensional manifolds) and morphisms (2-dimensional cobordisms), we get different data: respectively, 2-cocycles and 1-cocycles.

Definition 10. *The for a fixed finite group G and group 3-cocycle ω , the classical field theory is a symmetric monoidal 2-functor:*

$$(108) \quad \mathcal{A}_0(-)_G^\omega : \mathbf{3Cob}_2 \rightarrow \text{Span}(\mathbf{Gpd})^{U(1)}$$

which acts as follows:

- *Objects:* $\mathcal{A}_0(B)_G^\omega = (\mathcal{A}_0(B)_G, \tau_B(\omega))$
- *Morphisms:* $\mathcal{A}_0(S : B_1 \rightarrow B_2)_G^\omega = (\mathcal{A}_0(S)_G, \tau_S(\omega), i_1^*, i_2^*)$ (where the i_j are the inclusion maps of the B_j into S).
- *2-Morphisms:* $\mathcal{A}_0(M : S \rightarrow S')_G^\omega = (\mathcal{A}_0(M)_G, \tau_M(\omega), i^*, (i')^*)$, where again i and i' are inclusion maps of S and S' into M .

This definition implicitly makes the assertion that this is a symmetric monoidal 2-functor. The first thing to check is that it makes sense at all.

Lemma 2. *The construction for $\mathcal{A}_0(-)_G^\omega$ gives well-defined maps for objects, morphisms, and 2-morphisms into $\text{Span}(\mathbf{Gpd})^{U(1)}$.*

Proof. We need to check that the image of $\mathcal{A}_0(-)_G^\omega$ actually lies in $\text{Span}(\mathbf{Gpd})^{U(1)}$. It is well-known that transgression will yield cocycles (see e.g. [45]), so we need to verify the conditions (87) and (89) for those cocycles.

Suppose that $S : B \rightarrow B'$ is a cobordism, so that $\partial S = B \sqcup B'$, and applying $\mathcal{A}_0(-)_G^\omega$ we get the span:

$$(109) \quad \begin{array}{ccc} & (\mathcal{A}_0(S)_G, \tau_S(\omega)) & \\ \swarrow s & & \searrow t \\ (\mathcal{A}_0(B)_G, \tau_B(\omega)) & & (\mathcal{A}_0(B')_G, \tau_{B'}(\omega)) \end{array}$$

Then we want to verify that the cocycles are compatible, or in other words that $(s^*\theta_B)(t^*\theta_{B'})^{-1} = 1$. Restating this with the cocycle taking values in the additive group \mathbb{R}/\mathbb{Z} (since we want to express the value in terms of an integral):

$$(110) \quad s^*\theta_B - t^*\theta_{B'} = 0$$

But this is a 2-cocycle on $\mathcal{A}_0(S)_G$ given by:

$$\begin{aligned}
(111) \quad & (s^*\theta_B) - (t^*\theta_{B'}) \\
&= s^*\left(\int_B ev^*(\omega)\right) - t^*\left(\int_{B'} ev^*(\omega)\right) \\
&= \pi^*\left(\int_{\partial S} ev^*(\omega)\right)
\end{aligned}$$

(112)

Here, we are using the fact that the orientation on B and B' , the boundary components of S , are opposite, by convention and denoting by $\pi = s \otimes \bar{t}$ the projection map from $\mathcal{A}_0(S)_G$ to $\mathcal{A}_0(\partial S)_G$. So this says the difference of $s^*\theta_B$ and $t^*\theta_{B'}$ is $\pi^*(\tau_{\partial S}(\omega))$.

This is the pullback of a 2-cocycle on $Maps(\partial S, BG)$ to a 2-cocycle on $Maps(S, BG)$. Now suppose we evaluate it on a 2-chain Δ_2 in $Maps(S, BG)$, which we take to be a 2-simplex. (A similar proof would work for non-simplicial constructions of BG). Then:

$$\begin{aligned}
(113) \quad & \pi^*(\tau_{\partial S}(\omega))[\Delta_2] \\
&= \int_{\partial S \times \Delta_2} ev^*(\omega) \\
&= \int_{ev(\partial S \times \Delta_2)} (\omega)
\end{aligned}$$

This is an integral of ω on a 3-chain in BG which is one part of:

$$(114) \quad \partial(S \times \Delta_2) = (\partial S \times \Delta_2) \cup (S \times \partial \Delta_2)$$

So if we evaluate on the whole 3-chain, we have:

$$\begin{aligned}
(115) \quad & \int_{ev(\partial(S \times \Delta_2))} (\omega) \\
&= \int_{ev((\partial S \times \Delta_2) \cup (S \times \partial \Delta_2))} (\omega) \\
&= \int_{ev(\partial S \times \Delta_2)} (\omega) + \int_{ev(S \times \partial \Delta_2)} (\omega)
\end{aligned}$$

Now, since

$$(116) \quad \int_{ev(S \times \partial \Delta_2)} (\omega) = \tau_S(\omega)[\delta \Delta_2]$$

which is the evaluation of the 1-cocycle $\tau_S(\omega)$ on a boundary, this part is equal to 0. However, by Stokes' theorem:

$$(117) \quad \int_{ev(\partial(S \times \Delta_2))} (\omega) = \int_{ev(S \times \Delta_2)} \delta(\omega)$$

and since ω is a cocycle, this is again 0. Thus, we have

$$(118) \quad \pi^*(\tau_{\partial S}(\omega))[\Delta_2] = 0$$

so that finally $s^*\theta_B = t^*\theta_{B'}$ as required.

A similar argument with 1-cocycles and 0-cocycles holds at the level of 2-morphisms.

Now we have that $\mathcal{A}_0(-)_G^{\omega}$ gives well-defined maps of objects, morphisms, and 2-morphisms from $\mathbf{3Cob}_2$ into $\text{Span}(\mathbf{Gpd})^{U(1)}$. It remains to verify that it is

a symmetric monoidal 2-functor, by checking that the cocycle data respects this structure. \square

Now we want to verify that $\mathcal{A}_0(-)_G^\omega$ is indeed a symmetric monoidal 2-functor. Since we know $\mathcal{A}_0(-)_G : \mathbf{3Cob}_2 \rightarrow \text{Span}(\mathbf{Gpd})$ is a symmetric monoidal 2-functor, we only need to check that the cocycle data is as required.

As mentioned in Section 5.1 that the cocycle values themselves associated to manifolds and cobordisms with connection, by Picken's construction, define an HQFT. The properties proved in [38] amount to the fact that such an HQFT is a (symmetric) monoidal functor into vector spaces (1-dimensional in this case) from a category of manifolds and cobordisms which are equipped with a map into a target space X , which in this case is BG . Such HQFT are in 1-1 correspondence with gerbes on BG , which are determined by cocycles such as ω , which represent the curvature form for the gerbe.

Given $\mathcal{A}_0(-)_G^\omega$, the composition rule for $\text{Span}(\mathbf{Gpd})^{U(1)}$ discussed in the previous section gets a useful geometric interpretation. In the special case where 1-morphisms are manifolds without boundary, seen as cobordisms from the empty manifold to itself, the cocycle data for any groupoid of connections may be seen as a by a rank-1 embedded 2-dimensional TQFT with target BG in the sense of Picken (definition 4.1 of [38]). This may be understood as a (unitary) TQFT (or rather, HQFT, since manifolds and cobordisms are equipped with maps to BG) in which every vector space is just the 1-dimensional vector space \mathbb{C} . (Note that the adjective "unitary" is unnecessary for finite groups, since every element has finite order so all values are in fact roots of unity.)

Thus, one gets an element of $U(1)$ for each morphism of objects (i.e. between 1-manifolds equipped with connection - in other words, for a gauge transformation), by what Picken calls Z' (part of Z in our terminology), and an element of $U(1)$ for each cobordism by Picken's Z . Then the composition rule for cobordisms is Picken's gluing rule, which agrees with our composition rule (95) in that case. The point here is that one gets an extra contribution from the boundary B where two manifolds are being glued. This is explained in [38] in terms of transition functions (for a gerbe induced from the gerbe on BG classified by ω). Essentially, one must make a gauge transformation to ensure that connections on the two cobordisms being glued actually match at the boundary. We may also understand it by thinking of the gauge transformation identifying the different connections on B as a mapping cylinder: two copies of B with connections in different gauge, are identified with the ends of a cylinder $B \times I$. This has a nontrivial connection where the holonomies along the edge $b \times I$ for each point $b \in B$ conjugates an holonomy for a loop based at that point.

This is the extra morphism in the weak pullback contributing to this composition, as described in Section 5.2.

This is the idea behind the proof of the following.

Theorem 4. *The construction $\mathcal{A}_0(-)_G^\omega$ gives a symmetric monoidal 2-functor.*

Proof. First, we check that composition of 1-morphisms is preserved up to isomorphism. Suppose $S : B_1 \rightarrow B_2$ and $S' : B_2 \rightarrow B_3$ are cobordisms. Then

$$(119) \quad \mathcal{A}_0(S' \circ S)_G^\omega = (\mathcal{A}_0(S' \circ S)_G, \tau_{S' \circ S}(\omega), i_1^*, i_3^*)$$

On the other hand, since we have $\mathcal{A}_0(B_2)_G^\omega = (\mathcal{A}_0(B)_G, \theta_{B_2})$ and $\theta_{B_2} = \tau_{B_2}(\omega)$, it follows from (92) that:

$$(120) \quad \begin{aligned} \mathcal{A}_0(S')_G^\omega \circ S^\omega &= (\mathcal{A}_0(S')_G, \tau_{S'}(\omega), i_1^*, i_2^*) \circ (\mathcal{A}_0(S)_G, \tau_S(\omega), i_2^*, i_3^*) \\ &\cong (\mathcal{A}_0(S' \circ S)_G, \tau_{S'}(\omega) \cdot \tau_S(\omega) \cdot \tau_{B_2}(\omega), I_1^*, I_3^*) \end{aligned}$$

The composition of spans is just the weak pullback over $\mathcal{A}_0(B_2)_G$. The I_j are the inclusion maps of boundaries into the composite cobordism.

Since we know $\mathcal{A}_0(S' \circ S)_G \cong \mathcal{A}_0(S')_G \circ \mathcal{A}_0(S)_G$, it suffices to check that

$$(121) \quad \tau_{S' \circ S}(\omega) = \tau_{S'}(\omega) \cdot \tau_S(\omega) \cdot \tau_{B_2}(\omega)$$

under this identification. This is a 1-cocycle on the groupoid $\mathcal{A}_0(S' \circ S)_G$ of connections on the composite. That is, a $U(1)$ -valued function on gauge transformations which respects their composition.

Now, we are identifying the groupoid of connections on $S' \circ S$ with the weak pullback of $\mathcal{A}_0(S')_G$ and $\mathcal{A}_0(S)_G$ over $\mathcal{A}_0(B_2)_G$ (to which it is naturally equivalent). This means a connection on the whole space is determined by a pair of connections in $\mathcal{A}_0(S')_G \times \mathcal{A}_0(S)_G$ identified by a gauge transformation between the restrictions of the connections to B_2 . (That is, there is a “transition function” specifying the change of gauge when gluing the connections at B_2). A gauge transformation between two such objects is then a square of the form (100), and includes two gauge transformations from $\mathcal{A}_0(B_2)_G$ - the transition function for the gauge transformations. As in the discussion of Picken’s HQFT above, this gets a $U(1)$ -element assigned, just as if we glued using a mapping cylinder $B_2 \times I$ with a nontrivial connection. This factor is precisely $\tau_{B_2}(\omega)$ by this construction. The cocycle on $\mathcal{A}_0(S' \circ S)_G$ is then exactly this cocycle (pulled back through the equivalence with $\mathcal{A}_0(S')_G \circ \mathcal{A}_0(S)_G$).

A similar argument for 0-cocycles and gluing along 1-cocycles shows the composition of 2-morphisms is respected. Together these imply the preservation of all identity 1- and 2-morphisms.

It is straightforward to verify that $\mathcal{A}_0(A \sqcup B)_G^\omega = \mathcal{A}_0(A)_G^\omega \otimes \mathcal{A}_0(B)_G^\omega$. This is since in $\text{Span}(\mathbf{Gpd})$ this monoidal product is just the Cartesian product from \mathbf{Gpd} , and the cocycles simply multiply in $U(1)$. On the other hand, the transgressed cocycles are

$$(122) \quad \begin{aligned} \tau_{A \sqcup B}(\omega) &= \int_{A \sqcup B} ev^*(\omega) \\ &= \int_A ev^*(\omega) + \int_B ev^*(\omega) \\ &= \tau_A(\omega) + \tau_B(\omega) \end{aligned}$$

Since this sum is in \mathbb{R}/\mathbb{Z} , this is exactly what we expect. Likewise, the monoidal unit is preserved.

Thus, $\mathcal{A}_0(-)_G^\omega$ is a symmetric monoidal 2-functor. \square

5.4. The Twisted 2-Linearization Functor $\Lambda^{U(1)}$. We have suggested that the twisted classical theory behind the DW model takes values in $\text{Span}(\mathbf{Gpd})^{U(1)}$. Thus, we now want to understand the twisted analog of Λ , the quantization functor $\Lambda^{U(1)}$.

The essential point is that we use the representations of the twisted groupoid algebras such as $\mathbb{C}^{\theta_A}[A]$. This is the algebra of complex functions on morphisms of

A , with the “twisted” multiplication:

$$(123) \quad (F \star_A G)(f) = \sum_g F(g)G(g^{-1}f)\theta_A(g, g^{-1}f)$$

The sum is taken over all morphisms $g \in A$ whose target is the source of f : this is a twisted form of the usual convolution product. Given this notation, it is a standard fact that representations of this twisted algebra correspond to so-called “twisted representations” ρ of the groupoid itself, in which the usual composition rule is replaced by $\rho(g_1) \circ \rho(g_2) = \theta_A(g_1, g_2)\rho(g_1 \circ g_2)$. It is also possible to describe these as representations of a central extension of the groupoid (more usual in the case of a group). We will choose the first of these descriptions for convenience.

So 2-cocycles on objects twist the representation categories that appear as the output of Λ . The 1-cocycles, as we will see, twist the functors between them associated to spans, and the 0-cocycles twist the natural transformations.

In particular, 1-morphisms will involve restriction and induction of these twisted representations, for example pulling back along $s : (X, \alpha_X) \rightarrow (A, \theta_A)$ turns a representation of $\mathbb{C}^{\theta_A}[A]$ into a representation of $\mathbb{C}^{s^*\theta_A}[X]$. By preceding arguments, is the same as $\mathbb{C}^{t^*\theta_B}[X]$ since $s^*\theta_A = t^*\theta_B$.

Thus, it would be quite possible to repeat what Λ does to a span $(X, s, t) : A \rightarrow B$, which simply takes $t_* \circ s^*$: pull back a representation to X and push forward to B . However, if a cocycle α_X is present, we can “twist” this identification of $\mathbb{C}^{s^*\theta_A}[X]$ with $\mathbb{C}^{t^*\theta_B}[X]$ by α_X . This uses the map:

$$(124) \quad M_\alpha : \mathbb{C}^{s^*\theta_A}[X] \rightarrow \mathbb{C}^{t^*\theta_B}[X]$$

which takes $f : X \rightarrow \mathbb{C}$ to $\alpha \cdot f : X \rightarrow \mathbb{C}$. Of course, this is actually an automorphism of one algebra, since the twisting cocycles are actually equal by the condition (87). It is convenient, however, to represent M_α this way in what follows. A well-known and straightforward, but still crucial fact which we demonstrate here, is:

Proposition 4. M_α is an algebra isomorphism.

Proof. Clearly M_α is linear, so we check compatibility with the products. Suppose $F, G : X \rightarrow \mathbb{C}$, are thought of as elements of $\mathbb{C}^{s^*\theta_A}[X]$, with the product (123). Then applying M_α , at $f \in X$ we have:

$$(125) \quad \begin{aligned} (M_\alpha(F) \star_B (M_\alpha G))(f) &= \sum_g (\alpha \cdot F)(g)(\alpha \cdot G)(g^{-1}f)\theta_B(g, g^{-1}f) \\ &= \sum_g \alpha(g)F(g)\alpha(g^{-1}f)G(g^{-1}f)\theta_B(g, g^{-1}f) \\ &= \sum_g F(g)G(g^{-1}f)\delta\alpha(g, g^{-1}f)^{-1} \cdot \theta_A(g, g^{-1}f) \\ &= \sum_g F(g)G(g^{-1}f)\alpha(f) \cdot \theta_A(g, g^{-1}f) \\ &= \alpha(f) \cdot (F \star_A G)(f) \\ &= (M_\alpha(F \star_A G))(f) \end{aligned}$$

And indeed, since this map is plainly invertible with inverse $M_{\alpha^{-1}}$, this gives an isomorphism between the two algebras. \square

This isomorphism induces a specific (contravariant) isomorphism between the representation categories, by pre-composition:

$$(126) \quad M_\alpha^* : \text{Rep}(X, s^*\theta_A) \rightarrow \text{Rep}(X, t^*\theta_B)$$

We will use this in the construction for the 1-morphism map of $\Lambda^{U(1)}$.

Just as 2-cocycles twist the objects (representation categories) and 1-cocycles twist the 1-morphisms (functors), so the 0-cocycles will twist 2-morphisms (natural transformations). The untwisted Λ uses the unit and counit for the adjunction between induction and restriction functors of representations along groupoid homomorphisms. There is still an adjunction for algebra representations, so this part is much the same. However, Λ also uses the Nakayama isomorphism, as in (14). This is a canonical choice of isomorphism between the left and right adjoints to the restriction functor.

However, since $\mathbf{2Vect}$ is enriched in $\mathbf{Vect}_{\mathbb{C}}$, we can of course “twist” this natural isomorphism by a scalar that depends on a choice of object. This is exactly the role of the 0-cocycle (which is physically interpreted as the complex-valued “action” for the configuration of our QFT that object represents).

Definition 11. *The “twisted form” of the Nakayama isomorphism:*

$$(127) \quad N_{\beta_Y} : \sigma_* \circ (M_{\sigma^*\alpha_1})^* \circ \sigma^* \implies \tau_* \circ (M_{\tau^*\alpha_2})^* \circ \tau^*$$

which relates the (α -twisted) forms of the left and right adjunction which acts in each stage (object $y \in Y$) by:

$$(128) \quad N_{\beta_Y} : \bigoplus_{[y]|f(y) \cong x} \phi_y \mapsto \bigoplus_{[y]|f(y) \cong x} \frac{\beta_Y(y)}{\#\text{Aut}(y)} \sum_{g \in \text{Aut}(x)} g \otimes \phi_y(g^{-1})$$

This is just the same as the usual form, except for the factor of $\beta_Y(y)$. We note that this implicitly assumes that our spans of span maps commute exactly - as in [36], we might also need to incorporate an explicit isomorphism up to which the diagram commutes. We note also that by (89), the maps $(M_{\sigma^*\alpha_1})^*$ and $(M_{\tau^*\alpha_2})^*$ are in fact equal, so again this natural isomorphism is an automorphism.

Combining these twisted variants on the ingredients of Λ , we have the following:

Definition 12. *The 2-functor*

$$(129) \quad \Lambda^{U(1)} : \text{Span}(\mathbf{Gpd})^{U(1)} \rightarrow \mathbf{2Vect}$$

consists of the following assignments.

- **Objects:** $\Lambda^{U(1)}(A, \theta_A) = \text{Rep}(\mathbb{C}^{\theta_A}(A))$
- **Morphisms:** To a span $(X, \alpha_X, s, t) : (A, \theta_A) \rightarrow (B, \theta_B)$ define a 2-linear map:

$$(130) \quad \Lambda^{U(1)}(X, \alpha_X, s, t) = t_* \circ (M_{\alpha_X})^* \circ s^*$$

where $M_{\alpha_X} : \mathbb{C}^{s^*\theta_A}(X) \rightarrow \mathbb{C}^{t^*\theta_B}(X)$ is the isomorphism of these groupoid algebras induced by multiplication by α_X .

- **2-Morphisms:** to a 2-morphism $(Y, \beta_Y, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \Rightarrow (X_2, \alpha_2, s_2, t_2)$ assign the natural transformation:

$$(131) \quad \Lambda^{U(1)}(Y, \beta_Y, \sigma, \tau) = \epsilon_{L, \tau} \circ N_{\beta_Y} \circ \eta_{R, \sigma} : (t_1)_* \circ (M_{\alpha_1})^* \circ s_1^* \implies (t_2)_* \circ (M_{\alpha_2})^* \circ s_2^*$$

Remark 4. We have somewhat abused notation in order to write this in a balanced form. Strictly speaking, we have that:

$$(132) \quad \eta_{R,\sigma} : \mathrm{Id}_{\mathrm{Rep}(X_1, \alpha_1)} \Longrightarrow \sigma_* \circ \sigma^*$$

and similarly:

$$(133) \quad \epsilon_{L,\tau} : \tau_* \circ \tau^* \Longrightarrow \mathrm{Id}_{\mathrm{Rep}(X_2, \alpha_2)}$$

We have written them source and target, incorporating the multiplication operators M_{α_i} (and, though not written here, $M_{\sigma^* \alpha_1}$ and $M_{\tau^* \alpha_2}$). The point is just that

$$(134) \quad \sigma_* \circ (M_{\sigma^* \alpha_1})^* \circ \sigma^* \cong (M_{\alpha_1})^* \circ \sigma_* \circ \sigma^*$$

and similarly for τ .

It may help to note that the cocycles at each level play somewhat independent roles, *a priori* in this construction, though with our specific classical field theory $\mathcal{A}_0(-)_G^{\omega}$ they are closely related via transgression. This close connection may be an important part of the physical interpretation of this theory, and to ensuring we have a functor from $\mathbf{3Cob}_2$, but it is not essential to the “quantization functor” $\Lambda^{U(1)}$. The definition of $\Lambda^{U(1)}$ means we must have that $s^*(\theta_A)$ and $t^*(\theta_B)$ differ by the coboundary of α_X for the 2-linear map associated to a span to make sense (see the proof of the Theorem 5 below). However, the requirement α_X is a cocycle, hence has coboundary 0, so this is simply the requirement that $s^*(\theta_A)t^*(\theta_B)^{-1} = 1$ in the definition of $\mathrm{Span}(\mathbf{Gpd})^{U(1)}$. A similar remark applies to the 2-morphisms.

This means that the deep underlying relation between the θ , α , β cocycles in our ETQFT is a property of the classical field theory, not a requirement of the quantization functor. Indeed, part of the point of this factorization is that the quantization functor contributes little to an understanding of the system: it essentially looks at a specific representation in $\mathbf{2Vect}$ of structures already present in $\mathrm{Span}(\mathbf{Gpd})^{U(1)}$. To say this is a “representation” to say precisely the following, which was implicitly stated in the above definition:

Theorem 5. *The construction in Definition 12 determines a symmetric monoidal 2-functor*

$$(135) \quad \Lambda^{U(1)} : \mathrm{Span}(\mathbf{Gpd})^{U(1)} \rightarrow \mathbf{2Vect}$$

It is clear that $\mathrm{Rep}(C^{\theta_A}(A))$ is a 2-vector space, since it is the category of representations of a finite dimensional complex algebra on complex vector spaces. Similarly, the functorial constructions for 1- and 2-morphisms ensure that we must obtain 2-linear maps and natural transformations. We must show that these assemble into a symmetric monoidal 2-functor.

This is the twisted version of ([36], Thm. 5), though here we are also explicitly noting that the 2-functor is symmetric monoidal. Much of the proof is substantially the same as the untwisted case. We need to check several facts, so we will prove it as a series of lemmas, corresponding to lemmas and theorems shown in the untwisted case in [36]. The proofs are similar, so we will cite those at the appropriate place for brevity where there is significant overlap and show only the distinct new parts of the proofs.

Lemma 3. $\Lambda^{U(1)}$ *preserves composition of 1-morphisms up to isomorphism.*

Proof. Note that this is the twisted version of ([36], Thm. 3), which gives the corresponding isomorphism for the composites of spans in $\text{Span}(\mathbf{Gpd})$.

Suppose we are given two spans in $\text{Span}(\mathbf{Gpd})^{U(1)}$:

$$(136) \quad (X_1, \alpha_1, s_1, t_1) : (A, \theta_A) \rightarrow (B, \theta_B)$$

and

$$(137) \quad (X_2, \alpha_2, s_2, t_2) : (B, \theta_B) \rightarrow (C, \theta_C)$$

Then the composite is:

$$(138) \quad (X_2 \circ X_1, \alpha_2 \cdot \theta_B \cdot \alpha_1, s_1 \circ S, t_2 \circ T)$$

The cocycle is that given in (92), and $(X_2 \circ X_1, S, T)$ are the groupoid and maps in the weak pullback of the cospan (B, t_1, s_2) . It is shown in ([36], Thm. 3) that there is an isomorphism

$$(139) \quad \gamma : T_* \circ S^* \rightarrow (s_2)^* \circ (t_1)_*$$

for the untwisted representation categories. It suffices to show that a similar natural isomorphism between two functors

$$(140) \quad T_* \circ M_{\alpha_2 \cdot \theta_B \cdot \alpha_1} \circ S^*, (s_2)^*(t_1)_* : \text{Rep}^{s_1^* \theta_A}[X_1] \rightarrow \text{Rep}^{s_2^* \theta_B}[X_2]$$

First, note that the induction and restriction functors for twisted representations are given by the usual formulas for modules of rings, and that the twisted groupoid algebras are characterized as a direct sum of twisted group algebras. These are the group algebras for automorphism groups of the objects in the X_i and B , with multiplication twisted by the relevant 2-cocycles θ . For clarity, we will use the following notation for these algebras that appear in the restriction and induction formulas:

$$(141) \quad \mathbb{A}_{x_1} = \mathbb{C}^{s_1^* \theta_A}[\text{Aut}(x_1)]$$

$$(142) \quad \mathbb{A}_{x_2} = \mathbb{C}^{s_2^* \theta_B}[\text{Aut}(x_2)]$$

$$(143) \quad \mathbb{A}_{t_1(x_1)} = \mathbb{C}^{\theta_B}[\text{Aut}(t_1(x_1))]$$

$$(144) \quad \mathbb{A}_{x_1, x_2} = \mathbb{C}^{(s_1 \circ S)^* \theta_A}[\text{Aut}(x_1) \times_{\text{Aut}(t_1(x_1))} \text{Aut}(x_2)]$$

Note that the cocycles mentioned are necessarily equal to others - for instance, $s_1^* \theta_A = t_1^* \theta_B$, and so on.)

So this natural transformation can be expressed at a stage $x_1 \in X_1$ in terms of its action on a representation ρ . This is a linear map between spaces which are expressed as a direct sum over $x_2 \in X_2$, and in each such summand we have:

$$(145) \quad \gamma_{x_1}(F) : \mathbb{A}_{x_2} \otimes_{\mathbb{A}_{x_1, x_2}} \rho(x_1) \rightarrow \mathbb{A}_{t_1(x_1)} \otimes_{\mathbb{A}_{x_1}} \rho(x_1)$$

This is simply the twisted case of the usual formulas given as (91) and (92) in [36]. The isomorphism given there as (94) will still work in the twisted case. In the current notation, it acts in the following way. The algebra \mathbb{A}_{x_1, x_2} decomposes as a direct sum over all $g \in \text{Aut}(t_1(x_1))$ (since it is a group algebra of a fibre product). In the summand associated to g we define the isomorphism to act on the generators of $\mathbb{A}_{x_2} \otimes_{\mathbb{A}_{x_1, x_2}} \rho(x_1)$ by:

$$(146) \quad (k \otimes v) \mapsto s_2(k)g^{-1} \otimes v$$

which extends to the whole space. This is still well defined, though now uses the twisted multiplication in the group algebra \mathbb{A}_{x_2} . In particular, the underlying vector

spaces are identical to the untwisted case. The proof that this is an isomorphism is substantially the same as in the untwisted case. The main difference is that the cocycle θ_B enters into the twisting of the multiplication in \mathbb{A}_{x_1, x_2} and $\mathbb{A}_{t_1(x_1)}$. However, this twisting is compatible with the isomorphism γ and the algebra maps induced by the homomorphisms t_1 and s_2 into (B, θ_B) , so the proof is the same.

This then extends to an isomorphism between the two 2-linear maps

$$(147) \quad \begin{aligned} & \Lambda^{U(1)}((X_2, \alpha_2, s_2, t_2) \circ (X_1, \alpha_1, s_1, t_1)) \\ & = (T \circ t_2)_* \circ (M_{\alpha_1 \cdot \theta_B \cdot \alpha_2})^* \circ (S \circ s_1)^* \end{aligned}$$

and

$$(148) \quad \begin{aligned} & \Lambda^{U(1)}(X_2, \alpha_2, s_2, t_2) \circ \Lambda^{U(1)}(X_1, \alpha_1, s_1, t_1) \\ & = (t_2)_* \circ (M_{\alpha_2})^* \circ (s_2)^* \circ (t_1)_* \circ (M_{\alpha_1})^* \circ (s_1)^* \end{aligned}$$

□

Now we need the analogous fact for composition of 2-morphisms:

Lemma 4. $\Lambda^{U(1)}$ preserves vertical composition of 2-morphisms strictly and horizontal composition of 2-morphisms up to the isomorphism of Lemma 3.

Proof. This is the twisted analog of ([36], Lemma 4) and ([36], Lemma 5). The proofs are just the same except that we now have the factors N_β in (131). It thus suffices that the N_β are multiplicative under both horizontal and vertical composition of 2-morphisms.

For vertical composition, suppose we are given 2-morphisms

$$(149) \quad (Y, \beta_Y, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \rightrightarrows (X_2, \alpha_2, s_2, t_2)$$

and

$$(150) \quad (Y', \beta_{Y'}, \sigma', \tau') : (X_2, \alpha_2, s_2, t_2) \rightrightarrows (X_3, \alpha_3, s_3, t_3)$$

(Note that the current notation is different from that of [36], since here we use (σ, τ) instead of (s, t) , so that the structure maps for 2-morphisms are given by Greek letters and for 1-morphisms by Latin.)

Then by (95), we have:

$$(151) \quad \begin{aligned} & \Lambda^{U(1)}((Y', \beta_{Y'}, \sigma', \tau') \circ (Y, \beta_Y, \sigma, \tau)) \\ & = \Lambda^{U(1)}(Y' \circ Y, \beta_{Y'} \cdot \beta_Y \cdot \alpha_2, S \circ \sigma, T \circ \tau') \\ & = \epsilon_{L, (T \circ \tau)} \circ N_{\beta_{Y'} \cdot \beta_Y \cdot \alpha_2} \circ \eta_{R, (S \circ \sigma)} \end{aligned}$$

The terms appearing here are defined in and following Definition 12. On the other hand, we have:

$$(152) \quad \begin{aligned} & \Lambda^{U(1)}(Y', \beta_{Y'}, \sigma', \tau') \circ \Lambda^{U(1)}(Y, \beta_Y, \sigma, \tau) \\ & = \epsilon_{L, (\tau')} \circ N_{\beta_{Y'}} \circ \eta_{R, (\sigma')} \circ \epsilon_{L, (\tau)} \circ N_{\beta_Y} \circ \eta_{R, (\sigma)} \end{aligned}$$

(153)

This composite agrees with (151) by a similar argument to that for 1-morphisms. Namely, S and T are the maps for the weak pullback $(\sigma' \downarrow \tau)$, and these maps are compatible with twisted multiplication.

For horizontal composition, the proof is substantially the same as Lemma 5 of [36], except that factors of β appear in the sums, and twisting by θ_B makes the

maps of the pullback compatible with the twisted multiplication. The rest of the argument is substantially the same as for vertical composition.

So we have that composition of 2-morphisms is preserved. \square

Lemma 5. *The 2-functor $\Lambda^{U(1)}$ preserves symmetric monoidal structure.*

Proof. This is a straightforward observation:

$$(154) \quad \Lambda^{U(1)}((A_1, \theta_1) \otimes (A_2, \theta_2))$$

$$(155) \quad = \Lambda^{U(1)}(A_1 \times A_2, \theta_1 \cdot \theta_2)$$

$$(156) \quad = \text{Rep}^{\theta_1 \cdot \theta_2}[A_1 \times A_2]$$

$$(157) \quad = \text{Rep}(\mathbb{C}^{\text{theta}_1 \cdot \theta_2}[A_1 \times A_2])$$

$$(158) \quad \cong \text{Rep}(\mathbb{C}^{\text{theta}_1}[A_1] \otimes \mathbb{C}^{\theta_2}[A_2])$$

$$(159)$$

The isomorphism arises because the twisted multiplication acts independently in each factor. But this is generated by irreducible representations, and an irreducible representation of $\mathbb{C}^{\text{theta}_1}[A_1] \otimes \mathbb{C}^{\theta_2}[A_2]$ is a tensor product of irreps of $\mathbb{C}^{\text{theta}_1}[A_1]$ and $\mathbb{C}^{\theta_2}[A_2]$. So this is isomorphic to:

$$(160) \quad \Lambda^{U(1)}(A_1, \theta_1) \otimes \Lambda^{U(1)}(A_2, \theta_2)$$

$$(161) \quad = \text{Rep}^{\theta_1}[A_1] \otimes \text{Rep}^{\theta_2}[A_2]$$

with the tensor product the Deligne product of categories. (For 2-vector spaces, this looks just like the usual tensor product of vector spaces in terms of generators, up to isomorphism.)

The symmetry is preserved because all the tensor products essentially derive from \times for groupoids. Note that this argument is substantially the same for the twisted and untwisted cases. \square

Taking all these lemmas together we have the proof of Theorem 5. It is then immediate that this is an extension of our original 2-linearization 2-functor to the larger category, in the sense of the embedding noted in Corollary 2. That is:

Corollary 3. *The restriction of $\Lambda^{U(1)}$ to $\text{Span}(\mathbf{Gpd}) \subset \text{Span}(\mathbf{Gpd})^{U(1)}$, is isomorphic with Λ .*

Proof. This is a straightforward consequence of applying the definitions with trivial cocycles, and the fact that the representation category of a finite groupoid is canonically isomorphic to that of its groupoid algebra. \square

There is also a different special case, which is not immediately relevant to our ETQFT context, but which we will point out since it is immediate. This extends the fact that Λ restricted to $\text{Hom}(\mathbf{1}, \mathbf{1})$ reproduces Baez-Dolan groupoidification (shown in [36]). The new special case incorporates the $U(1)$ -groupoids of [34]. Then a $U(1)$ -groupoid span (or “stuff operator” in the sense of [34]) is simply a nontrivial 2-morphism of $\text{Span}(\mathbf{Gpd})^{U(1)}$ in $\text{Hom}((\mathbf{1}, 1), (\mathbf{1}, 1))$ between two 1-morphisms with trivial cocycles. (Moreover, [34] only considered the case where the central object of these spans are always the groupoid of finite sets and bijections. This is not essentially finite, as in the present case, but provided we restrict to situations where all sums converge, the same ideas apply.)

Finally, as in the untwisted case, the matrix representations of the 2-linear maps are straightforward to describe:

Proposition 5. *Given a 1-morphism:*

$$(162) \quad (X, \alpha, s, t) : (A, \theta_A) \rightarrow (B, \theta_B)$$

the a 2-linear map $\Lambda^{U(1)}(X, \alpha, s, t)$ has matrix representation whose components are:

$$(163) \quad \Lambda^{U(1)}(X, \alpha_x, s, t)_{\rho, \phi} = \text{Hom}_{\text{Rep}(\mathbb{C}^{s^* \theta_A X})}((s^* \rho), ((M_\alpha)^* t^* \phi))$$

Proof. As in the untwisted case, the main point here is Frobenius reciprocity, in this case for representations of algebras. The operation of pulling back a representation ϕ of B ($\mathbb{C}^{\theta_B}[B]$) is adjoint to the operation of pushing-forward a representation of $\mathbb{C}^{t^* \theta_B}[X]$.

As in the untwisted case, the $\Lambda^{U(1)}(X, \alpha, s, t)$ takes a θ_A -twisted representation of A , and pulls back to X , then pushes forward to B . The difference is that we apply the map M_α between these steps. By Frobenius reciprocity, we have the intertwiner space:

$$(164) \quad \begin{aligned} & \text{Hom}_{\text{Rep}(\mathbb{C}^{\theta_B}[B])}(t_* \circ M_\alpha^* \circ s^* \rho, \phi) \\ &= \text{Hom}_{\text{Rep}(\mathbb{C}^{t^* \theta_B}[X])}(M_\alpha^* \circ s^* \rho, t^* \phi) \\ &= \text{Hom}_{\text{Rep}(\mathbb{C}^{s^* \theta_A}[X])}(s^* \rho, (M_\alpha)_* \circ t^* \phi) \end{aligned}$$

□

So given twisted irreps ρ and ϕ of A and B , (i.e. irreps of the twisted groupoid algebras), we find a component in a matrix for a 2-linear map as an intertwiner space between the pulled-back representations $s^* \rho$ and $t^* \phi$. These are twisted reps of X , but a priori we have that $s^* \rho$ is twisted by the cocycle $s^*(\theta_A)$, and $t^* \phi$ is twisted by $t^*(\theta_B)$.

We note that in principle, given two representations twisted by different cocycles, we would take the vector space of global sections of:

$$(165) \quad (s^* \rho) \otimes (t^* \phi)$$

This is a vector bundle on the objects of X , and when the cocycles coincide, it corresponds to the usual hom space. The "bar" means we take the dual representation of $s^* \rho$, which is a $(s^* \theta_A)^{-1}$ -twisted rep of X , so the tensor product (165) is a $(s^* \theta_A)^{-1} \times (t^* \theta_B)$ -twisted rep of X .

This suggests that a further generalization of our $\text{Span}(\mathbf{Gpd})^{U(1)}$ may be possible in which the condition (87) can be weakened. We might only require the α be a cochain, and that $s^* \theta_A$ and $t^* \theta_B$ should differ by the coboundary of a cochain α . This would ensure that M_α still induces an algebra isomorphism, but not necessarily an automorphism. If the condition were even weaker, the spaces of sections (165) would not correspond to intertwiner spaces. A similar generalization should be possible for condition (89), so that the coboundary of β_Y gives the difference between pullbacks of α_1 and α_2 .

This generalization, however, is not necessary for our construction of this ETQFT, so we will not consider it further here.

5.5. Twisted ETQFT. Finally, our main result asserts that the Dijkgraaf-Witten model can be understood as factorized into the classical field theory and the 2-linearization “quantization functor” we have just defined. The theory itself, as described originally by Dijkgraaf and Witten [13], is given in more explicit detail by Freed and Quinn [17], particularly in the situation of manifolds with boundary, which is the case where an ETQFT is most appropriate. This is the description to which we will refer here when speaking of the DW model. In particular, much of the description is in section 4 of [17] which describes its construction as a modular functor. We will describe how it is derived from the 2-functor we have given as our ETQFT.

Theorem 6. *Given a finite gauge group G and 3-cocycle $\omega \in Z^3(BG, U(1))$, the symmetric monoidal 2-functor*

$$(166) \quad Z_G^\omega = \Lambda^{U(1)} \circ \mathcal{A}_0(-)_G^\omega : \mathbf{3Cob}_2 \rightarrow \mathbf{2Vect}$$

reproduces the Dijkgraaf-Witten model with twisting cocycle ω .

Proof. First, we note that the DW model as described in [17] assigns a Hilbert space to each manifold with boundary. We will think of this manifold as a cobordism, and it will be necessary to describe the Hilbert space as a 2-linear map assigned to it.

To the boundary in such a case, the DW model assigns a collection of labels. These are irreducible representations of a certain algebra, which reduces to the case of the algebra assigned to a circle:

$$(167) \quad A^* = \bigoplus_{[T]} L_{[T]}^*$$

This is a direct sum over $[T]$, the distinct conjugacy classes in G , which is to say, the isomorphism classes of objects of $\mathcal{A}_0(S^1)_G$. The algebras L^* and connecting isomorphisms between them form a line bundle over the space of $[T]$, which is classified by a cohomology class given by the transgression of the form there called $\hat{\alpha}$, and here ω in $H^3(BG, \mathbb{R}/\mathbb{Z})$. (This is [17] Proposition 3.14).

The properties of the cocycles given in ([17], Prop. B.1 and B.10) agree with the cocycles used in our 2-linear map. In particular, the horizontal composition of 2-morphisms agrees with the gluing rule on a partial boundary of (Prop B.2 and B.11). The cocycles given there agree with the transgression rule given in (107) above and the trace over a tensor product is a description of horizontal composition which does not distinguish incoming and outgoing faces. So the cocycles $\beta_{\mathcal{A}_0(M)_G}$ are as we expect for a 3-dimensional cobordism M . We have summarized these functorial properties by the observation of 5.1 that the assignment of cocycles is an HQFT.

Now the algebra structure of $L_{[T]}^*$ is such that its unit vectors form a central extension of the centralizer of $[T]$ (that is, $Aut([T])$ in the sense of the groupoid $\mathcal{A}_0(S^1)_G$). The central extension is classified by the cocycle just mentioned. In our terminology, this says precisely that $L_{[T]}^*$ itself is the summand of $\mathbb{C}^{\tau_{S^1}(\omega)}[\mathcal{A}_0(S^1)_G]$ associated to $[T]$. Thus,

$$(168) \quad A^* \cong \mathbb{C}^{\tau_{S^1}(\omega)}[\mathcal{A}_0(S^1)_G]$$

But our Z_G^ω assigns the circle the representation category of this algebra, which recovers the label set assigned by the DW model.

Next, the DW model assigns a Hilbert space to each manifold with boundary Y (in the following we use the notation of [17], Section 3). We will understand this to be a cobordism relating its boundary components. Thus, this Hilbert space is to be understood as a 2-linear map. As a Hilbert space, the $E(Y, \lambda)$ are given as:

$$(169) \quad E(Y) = L^2(\overline{\mathcal{C}'_Y}, \overline{\mathcal{L}_Y})$$

That is, it is the space of (square-integrable, which condition is vacuous in the finite case) sections of a certain line bundle \mathcal{L}_Y over the space \mathcal{C}'_Y of flat connections (i.e. bundles with flat connections) on Y , which we would describe as the space of objects of $\mathcal{A}_0(Y)_G$. This bundle assigns a 1-dimensional space to each such object, and to each homotopy of the classifying maps of these flat bundles (i.e. to each morphism f of $\mathcal{A}_0(Y)_G$) an isomorphism of these lines, given by (3.4) of [17]. This isomorphism incorporates a factor which comes from an integral of $\hat{\alpha}$, or in our terms ω . This factor is just the value of $\tau_Y(\omega)$ on f .

A decomposition of the space of sections $E(Y)$ as a direct sum is given in ([17], sec. 4):

$$(170) \quad E(Y) \cong \bigoplus_{\lambda} E(Y, \lambda) \otimes E_{\lambda}$$

Here, the λ run over all labels for the boundary: this is a product of labels $\lambda = (\lambda_i)_i$ over all boundary components $(\partial Y)_i$. The representations E_{λ} associated to the whole boundary are therefore of the form $\otimes_i E_{\lambda_i}$. By the duality of Hom and \otimes , these are isomorphic to the intertwiner spaces given in (163).

The above decomposition amounts to treating $E(Y)$ as a module for A^* for the algebra associated to ∂Y , which acts on the E_{λ} , or rather as a bimodule for the algebras A^* for the source and target objects (taking the conjugate algebra when changing orientation, hence turning a left action into a right action). Frobenius reciprocity then ensures that taking a tensor product with this bimodule will act as multiplying by the matrix (163). This gives an interpretation of $E(Y)$ as the 2-linear map $Z_G^{\omega}(Y)$.

The DW model then assigns a map between these Hilbert space $E(Y)$ for each cobordism between manifolds Y and Y' . We further note that the inner product on this space, as a space of sections, is twisted by the cocycle α , which is accomplished precisely by the inclusion of the map M_{α} in our construction of the 2-linear map $Z_G^{\omega}(Y)$.

Finally we check that $Z_G^{\omega} = \Lambda^{U(1)} \circ \mathcal{A}_0(-)_G^{\omega}$ gives the data of the twisted DW model for 2-morphisms M of $\mathbf{3Cob}_2$, which are understood as 3-dimensional cobordisms of manifolds with boundary. In [17] they are described as manifolds with corners.

Part of this proof is substantially the same as that of Theorem 2, which shows in the untwisted case with empty boundary that our formula reproduces the (unnumbered) formula directly following ([17], 5.14). That formula uses the “mass” of a connection on (there described as a “representation” of $\pi_1(M)$ into the gauge group - though they denote the manifold by Y), which is just the groupoid cardinality $\frac{1}{|Aut([A])|}$ for a class $[A]$ of connections (denoted there by γ), as in our formula. This gives the measure used in the integrals over the space of connections, as we expect.

Finally, the explicit calculations of amplitudes in [17] are generally contractions of the 2-linear maps we obtain. Moreover, they are converted to amplitudes from linear operators between representation spaces by converting representations to

characters, taking the trace. Thus, since the trace of the identity for a representation ρ is $\dim(\rho)$, the formulae there contain factors of $\dim(E_\lambda)$, where E_λ is the representation space for a representation on the whole boundary (for us, the tensor product of the representations determining a given component of the natural transformation). So finally the computations of amplitudes such as ([17], 5.4) for the torus are precisely the result of applying this procedure to the natural transformations from $\Lambda^{U(1)}$.

We conclude that the Dijkgraaf-Witten model for manifolds with corners as presented in [17] can be recovered from our Z_G^ω . \square

6. CONCLUSION

In this paper, our goal has been to give a concrete description of the quantization functor which plays a role in the Dijkgraaf-Witten model. This is consistent with the program of Freed-Hopkins-Lurie-Teleman [16], in which topological quantum field theories are described in terms of a factorization into two parts. The first part, the classical field theory, takes values in groupoids. The second part assigns algebraic data to groupoids - in particular, k -vector spaces, or indeed k -algebras in an appropriate sense, to the groupoids associated to codimension- k manifolds.

The point is that the groupoids represent the moduli space for the field configurations of the classical theory. As we have seen, the full functor, to describe the DW model in its complete form, must incorporate the effect of data from groupoid cohomology.

One purpose of studying the quantization functor separately is that we hope to gain some understanding of the nature of the quantization process. Quantization is well-studied in the situation of a process (in good situations, a functor) taking classical configuration spaces to quantum Hilbert spaces. The higher-categorical k -vector spaces are less commonly used in the physical context and ETQFT gives a sufficiently simple, yet nontrivial, setting in which to study this aspect of quantization. What our functors Λ and its twisted version $\Lambda^{U(1)}$ illustrate is that this process can be described in terms of a simple, quite universal process once we understand the category $\text{Span}(\mathbf{Gpd})$, or its twisted version $\text{Span}(\mathbf{Gpd})^{U(1)}$.

In particular $\Lambda^{U(1)}$ is an extension of the very natural "2-linearization" process Λ , which is entirely canonical. Groupoids are taken to their representation categories. The morphisms (spans) are taken to 2-linear maps constructed naturally from induction and restriction functors. The 2-morphisms (spans of spans) are taken to natural transformations constructed naturally from the unit and counit for the adjunctions between these functors. This is an entirely canonical process generalizing the straightforward "pull" and "push" of functions through spans of sets which gives (natural number valued) matrix multiplication. Thus, the quantization functor is simply giving a canonical representation of $\text{Span}(\mathbf{Gpd})$, which is then in some sense the fundamental setting.

One important fact in the untwisted case, if we take the representation category $\Lambda(A)$ to be concrete, with its natural "underlying vector space" functor into \mathbf{Vect} , we have a Tannaka-Krein reconstruction theorem. That is, this 2-vector space (and the forgetful functor into \mathbf{Vect}) allows the groupoids (objects of $\text{Span}(\mathbf{Gpd})$) to be reconstructed completely. At the level of morphisms, and particularly 2-morphisms, however, we do lose information. This is easy to see in the special case of $\text{Hom}(\mathbf{1}, \mathbf{1})$, where $\Lambda^{U(1)}$ restricts to give groupoidification. Here, spans

of groupoids, as morphisms, are taken to linear maps whose components detect only *groupoid cardinalities* of spans. This does not determine the groupoids up to isomorphism. So in particular, the quantization functor is not faithful, and forgets information about the classical category as part of the “sum over histories” which defines the 2-morphisms.

The motivation for using $\text{Span}(\mathbf{Gpd})$ and its twisted extension is how it reflects physically important aspects of the quantum field theory. The objects are groupoids because the moduli problem for gauge theory, like many other geometric structures, has symmetries which are not seen in a topological space of field configurations. The quantization functor, our $\Lambda^{U(1)}$, is able to retain this information about symmetry, since it assigns the representation category to such groupoids. This characterizes them up to Morita equivalence. In general, systems whose configuration spaces are represented by Morita-equivalent groupoids are “physically indistinguishable”.

However, rather than working in the 2-category \mathbf{Gpd} , we expand it to consider $\text{Span}(\mathbf{Gpd})$. We have noted some work ([23], [26]) on such span categories generally. One important fact is that in general $\text{Span}(\mathbf{C})$ for a (1-)category is a universal (bi)category containing a copy of \mathbf{C} , for which every morphism has a (two-sided) adjoint. In the case \mathbf{C} is a bicategory, such as \mathbf{Gpd} , our construction also gives adjoints for 2-morphisms (in [23], one gets a monoidal tricategory, which we have made into a bicategory by taking 2-morphisms as mere equivalence classes of spans of span maps). This construction of “adjoining adjoints” is somewhat analogous to localization, which forces morphisms to be invertible. Instead, we force morphisms to be adjointable. This is the key feature captured in $\text{Span}(\mathbf{Gpd})$, and is also a key characteristic of the linear and 2-linear category.

The physical significance of adjointability is that if a morphism describes a process by which a system evolves, its adjoint is the same process with the reversed time-sense. In the ETQFT case, the cobordism category suggests that we should think of 2-morphisms as “time evolution” in this sense. The 1-morphisms then describing a space with boundary as linking its boundary components, and the adjoint simply reverses the sense of input and output boundary components.

In the twisted case, we must expand this setting to $\text{Span}(\mathbf{Gpd})^{U(1)}$, but this behaves much like $\text{Span}(\mathbf{Gpd})$ except that the groupoids carry extra cocycle information. This information is the higher-categorical extension of the Lagrangian functional, which is simply the 0-cocycle associated to 2-morphisms. This fits the approach of [16], in which the cocycle ω on BG , and the gerbe it classifies, is taken to be the true physical setting for the action. The transgressions to the moduli spaces for connections on manifolds of different dimensions are then particular manifestations of this action.

In subsequent work, it may be of interest to consider whether this larger bicategory $\text{Span}(\mathbf{Gpd})^{U(1)}$, or perhaps a weaker variation, has some important universal properties analogous to those of $\text{Span}(\mathbf{Gpd})$. For now, it is sufficient to observe that it is the natural target for the classical field theory of the DW model, and likely other interesting toy physics models relevant to TQFT. A subsequent paper by the author with Derek Wise will consider an analogous construction with compact Lie groups and give an explicit construction of a generalization of Λ and $\Lambda^{U(1)}$ which applies in the infinite setting.

7. ACKNOWLEDGEMENTS

The author would like to thank the referees of earlier versions for important suggestions, as well as John Baez, Dan Christensen, Derek Wise, Thomas Nikolaus, Jamie Vicary and Alex Hoffnung for valuable discussions.

REFERENCES

- [1] ADEM, A., COHEN, F. R., AND TORRES-GIESE, E. Commuting elements, simplicial spaces, and filtrations of classifying spaces. [arXiv:0901.0137](https://arxiv.org/abs/0901.0137), 2009.
- [2] ATIYAH, M. *Topological Quantum Field Theory*. Cambridge University Press, 1990.
- [3] BAEZ, J. Higher-dimensional algebra ii: 2-hilbert spaces. *Adv. Math.*, 127 (1997), 125–189. [arXiv:q-alg/9609018](https://arxiv.org/abs/q-alg/9609018).
- [4] BAEZ, J., AND DOLAN, J. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.* 36 (1995), 6073–6105. [arXiv:q-alg/9503002](https://arxiv.org/abs/q-alg/9503002).
- [5] BAEZ, J. C., HOFFNUNG, A. E., AND WALKER, C. D. Groupoidification made easy. Preprint at [arXiv:0812.4864](https://arxiv.org/abs/0812.4864).
- [6] BAEZ, J. C., HOFFNUNG, A. E., AND WALKER, C. D. Higher dimensional algebra vii: Groupoidification. [arXiv:0908.4305](https://arxiv.org/abs/0908.4305), 2009.
- [7] BARTLETT, B. Categorical aspects of topological quantum field theories. Master’s thesis, Utrecht University, 2005. [arXiv:math/0512103](https://arxiv.org/abs/math/0512103).
- [8] BENSON, D. *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*. No. 30 in Cambridge studies in advanced mathematics. Cambridge University Press, 1995.
- [9] BROWN, R. *Groupoids and Topology*. Booksurge, 2006.
- [10] CHENG, E., AND LAUDA, A. Higher-dimensional categories: An illustrated guidebook. Preprint at <http://www.dpmms.cam.ac.uk/~elgc2/guidebook/>.
- [11] CHORNY, B. Brown representability for space-valued functors. <http://arxiv.org/abs/0707.0904>, 2007.
- [12] DAWSON, R. J. M., PARÉ, R., AND PRONK, D. A. Universal properties of span. *Theory and Applications of Categories* 13, 4 (2004), 61–85.
- [13] DIJKGRAAF, R., AND WITTEN, E. Topological gauge theory and group cohomology. *Commun. Math. Phys.* 129 (1990). <http://math.ucr.edu/home/baez/qg-winter2005/group.pdf>.
- [14] DUBUC, E. J. *Kan extensions in enriched category theory*. Lecture Notes in Mathematics 145. Springer-Verlag, Berlin-New York, 1970.
- [15] FREED, D. S. Higher algebraic structures and quantization, 1993. [arXiv:hep-th/9212115](https://arxiv.org/abs/hep-th/9212115).
- [16] FREED, D. S., HOPKINS, M. J., LURIE, J., AND TELEMAN, C. Topological quantum field theories from compact lie groups. [arXiv:0905.0731](https://arxiv.org/abs/0905.0731), 2009.
- [17] FREED, D. S., AND QUINN, F. Chern simons theory with finite gauge group. *Commun. Math. Phys.*, 156 (1993).
- [18] GRANDIS, M. Higher cospans and weak cubical categories (cospans in algebraic topology i). *Theory and Applications of Categories* 18, 12 (2007), 321–347.
- [19] GRANDIS, M. Cubical cospans and higher cobordisms (cospans in algebraic topology iii). *Journal of Homotopy and Related Structures* 3, 1 (2008).
- [20] GREENLEES, J. Rational mackey functors for compact lie groups i. *Proc. London Math. Soc.* 3, 76 (1998), 549–578.
- [21] HIGGINS, P. J. *Categories and Groupoids*. No. 32 in Van Nostrand Reinhold Mathematical Studies. Van Nostrand Reinhold Company, London, 1971.
- [22] HOFFNUNG, A. The hecke bicategory. [arXiv:1007.1931](https://arxiv.org/abs/1007.1931).
- [23] HOFFNUNG, A. Spans in 2-categories: A one-object tetracategory. Unpublished draft at http://mysite.science.uottawa.ca/hoffnung/papers_files/spans.pdf.
- [24] KAPRANOV, M., AND VOEVODSKY, V. 2-categories and Zamolodchikov tetrahedron equations. *Proc. Symp. Pure Math* 56 Part 2 (1994), 177–260.
- [25] KASHIWARA, M., AND SCHAPIRA, P. *Categories and Sheaves*. Grundlehren der mathematischen Wissenschaften 332. Springer Verlag, 2000.
- [26] KENNEY, T., AND PRONK, D. The span profunctor and generalised span constructions. Unpublished draft.

- [27] KERLER, T., AND LYUBASHENKO, V. V. *Non-semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*. Lecture Notes in Mathematics, 1765. Springer Verlag, 2001.
- [28] LACK, S. A 2-categories companion. In *Towards Higher Categories*, J. C. Baez and J. May, Eds., no. 152 in The IMA Volumes in Mathematics and its Applications. Springer, 2010, pp. 105–191.
- [29] LAUDA, A. D. Frobenius algebras and ambidextrous adjunctions. *Theory and Applications of Categories* 16, 4 (2006), 84–122.
- [30] LAWRENCE, R. Triangulation, categories and extended field theories. In *Quantum Topology*, R. Baadhio and L. Kaufman, Eds. World Scientific, Singapore, 1993, pp. 191–208.
- [31] LURIE, J. On the classification of topological field theories. [arXiv:math.CT/0905.0465](https://arxiv.org/abs/math/0905.0465).
- [32] MACLANE, S., AND MOERDIJK, I. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer Verlag, 1992.
- [33] MOERDIJK, I. étale groupoids, derived categories, and operations. In *Groupoids in analysis, geometry, and physics* (2001), R. Amer. Math. Soc., Providence, Ed., no. 282 in *Contemp. Math.*, pp. 101–114.
- [34] MORTON, J. C. Categorified algebra and quantum mechanics. *Theory and Applications of Categories* 16 (2006), 785–854. <http://www.tac.mta.ca/tac/volumes/16/29/16-29.pdf>.
- [35] MORTON, J. C. Double bicategories and double cospans. *Journal of Homotopy and Related Structures* 4, 1 (2009), 389–428. Preprint at [arXiv:math/0611930](https://arxiv.org/abs/math/0611930).
- [36] MORTON, J. C. Groupoids and 2-vector spaces. *Applied Categorical Structures* (2010). Preprint at [arXiv:0810.2361](https://arxiv.org/abs/0810.2361), final version DOI:10.1007/s10485-010-9225-0.
- [37] PANCHADCHARAM, E. *Categories of Mackey Functors*. PhD thesis, Macquarie University, 2006.
- [38] PICKEN, R. F. Tqft’s and gerbes. *Alg. Geom. Top.* 4 (2004), 243–272.
- [39] PORTER, T. The crossed menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology. <http://ncatlab.org/nlab/files/menagerie10a.pdf>.
- [40] SCHOMMER-PRIES, C. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, University of California, Berkeley, 2009. <http://sites.google.com/site/chrissschommerpriesmath/>.
- [41] STERNBERG, S. *Group theory and physics*. Cambridge University Press, 1994.
- [42] THURSTON, W. *Three-Dimensional Geometry and Topology*. Princeton University Press, 1997.
- [43] TURAEV, V. *Quantum Invariants of Knots and 3-Manifolds*. No. 18 in De Gruyter Studies in Mathematics. Walter de Gruyter, 1994.
- [44] TURAEV, V. *Homotopy Quantum Field Theory*. No. 10 in Tracts in Mathematics. European Mathematical Society, 2010.
- [45] WILLERTON, S. The twisted drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.* 8, 3 (2008), 1419–1457.
- [46] YETTER, D. Categorical linear algebra - a setting for questions from physics and low-dimensional topology. Kansas State University, preprint. <http://math.ucr.edu/home/baez/yetter.pdf>.
- [47] YETTER, D. Topological quantum field theories associated to finite groups and crossed g -sets. *J. Knot Theory Ramifications* 1, 1 (1992), 120.

CAMGSD, INSTITUTO SUPERIOR TECNICO, UNIVERSIDADE TECNICO DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

E-mail address: jmorton@math.ist.utl.pt