GROUPOIDS AND CATEGORIFICATION IN PHYSICS

Abstract. This paper will describe one application of category theory to physics, known as "categorification". We consider a particular approach to the process, "groupoidification", introduced by Baez and Dolan, and interpret it as a form of explanation in light of a structural realist view of physical theories. This process attempts to describe structures in the world of Hilbert spaces in terms of "groupoids" and "spans" . We consider the interpretation of these ingredients, and how they can be used in describing some simple models of quantum mechanical features. In particular, we show how the account of entanglement in this setting is consistent with a a relational understanding of quantum states.

1. Introduction

The study of categories, which began in the 1940's, makes contact with most areas of mathematics, and applications to physics have begun to appear because categories provide new tools for looking at mathematical models of physical situations. In this paper, we consider the concept of "categorification", a term coined by Frenkel and Crane [5, 6], which refers to finding structures described in the language of categories which are analogous to structures defined in terms of sets. In particular, we will consider it as a form of explanation.

One long-standing claim, relevant to our goal here, is that category theory gives a "structural" view of mathematical entities, in terms of their relationship to other entities. This is contrasted with the view given by set theory, in which objects are understood in terms of the elements of which they are built[1]. Baez and Dolan [8] explain the motivation this way:

> One philosophical reason for categorification is that it refines our concept of 'sameness' by allowing us to distinguish between isomorphism and equality... Even more importantly, two objects can be the same in more than one way, since there can be different isomorphisms between them. This gives rise to the notion of the 'symmetry group' of an object: its group of automorphisms.

Thus, rather than equality, structures built of sets are more meaningfully defined up to isomorphism: for instance, the physical meaning of a configuration space does not depend on the exact identity of the points in its mathematical representation, but only their structure. For structures built of categories, isomorphism is again too strict, but there is corresponding weaker idea of *equivalence*.

A set is an elementary mathematical entity, with a single primitive concept, namely that of *element*, and the *membership* of elements in sets. A set S is determined precisely by its elements. The ontology for categories is slightly more complicated. A category is a structure with two sorts of primitive entities, namely

[1]The potential relevance of category theory, especially in its guise as "higher dimensional algebra", to the philosophy of mathematics is rather broader, as suggested by Corfield [4], but this distinction is enough for now.

objects and *arrows* between objects. We emphasize that these are primitive notions in category theory, along with the notion of the *source* and *target* object of an arrow, and the *composite* of arrows. This may not always be possible: arrows f and g can be composed to get a arrow $f \circ g$ as long as the source object of f is the same as the target object of g.

Categorification is not a systematic process, and as a form of explanation we should not expect it to be: given an arbitrary collection of facts, there is no standard way to find an explanation for them. We may hope, though, that we will recognize such an explanation when we find it. The most rigorous way to define a categorification of a set-based structure is to make one recognizable by the fact that the original structure appears when we "decategorify" it.

A special type of categorification is called "groupoidification" [2], since a groupoid is a special type of category. It again amounts to reversing a procedure called "degroupoidification". Two fundamental physical ideas which account for its relevance to physics are: *spaces of histories*; and *physical symmetry*. We understand this as a mode of explanation for structures built from Hilbert spaces and linear maps, and as such it is particularly relevant to quantum mechanics. It has a special role for the "sum over histories" of Lagrangian quantum mechanics.

Our current goal is not to claim that this particular mode of categorification gives a correct explanation. Rather, we want to show how groupoidification accounts for certain features of quantum mechanics, by a kind of explanation which is compatible with a structural realist view of physical theories.

2. Categorification and Groupoidification

2.1. **Categorification and Structural Realism.**

We will begin with a familar example, the explanation of arithmetic (natural numbers and their operations) in terms of operations in the category of finite sets. This reverses the act of *counting*, which is thus a form of decategorification.

There is a category **FinSet**, whose objects are finite sets and in which an arrow with source A and target B is a function $f : A \to B$. As far as this category is concerned, we take the arrows to be as fundamental as the sets themselves.

This category has interesting properties: it has products and "coproducts". These can be given abstract definitions, but concretely they are respectively the Cartesian product, and the disjoint union, of sets. Thus $(\mathbb{N}, +, \times)$, a structure defined with sets, can be "explained" as a decategorification of $(\mathbf{FinSet}, \sqcup, \times)$, where the operations are determined by certain universal properties.

Thus, the decategorification operation "taking the set of isomorphism classes" takes any category to a set. It takes the category of (relatively concrete) finite sets to the set of (relatively formal, abstract) numbers. Its effect on a given set is just to count its elements. Arithmetic operations like "$3 \times 4 = 12$" are then a shorthand way of describing structural operations on sets.

Categorification is a form of explanation in exactly this sense. It is consistent with a structural realist view of the process of refining scientific theories, in that there may be many categorical structures which explain the same set-based structure. Their objects may be internally very different, yet the structures they explain are preserved by maps between them.

Take the category **FinVect**, whose objects are finite-dimensional vector spaces, and whose arrows are linear maps. Objects are isomorphic when they have the same dimension, so the process of taking isomorphism classes is just a map into $\mathbb{N}$, now interpreted as "taking dimensions" rather than "counting". The tensor product $\otimes$ and direct sum $\oplus$ give the operations $\times$ and $+$ on isomorphism classes. So $(\mathbf{FinVect}, \oplus, \otimes)$ is another categorification of $(\mathbb{N}, +, \times)$, with a different interpretation.

The "real" (i.e. structural) distinctions between the two categories are captured precisely by the arrows which were discarded by decategorification. Without referring to the internal makeup of the objects, we can distinguish them by the groups of automorphisms of the two objects corresponding to the number n. In Set, this is the symmetric group S_n, and in **FinVect**, it is the general linear group $GL(n)$.

So we have two potential "explanations" of $\mathbb{N}$. They are different, but there is a map from **FinSet** to **FinVect**, which takes a set S to its "linearization", the free vector space $L(S) = \mathbb{C}[S]$. This preserves the properties of "being a product" and "being a sum" in the sense relevant to each category. The existence of the structure-preserving maps between these explanations (technically, functors between categories) accounts for the fact that both categorify $\mathbb{N}$. The structural realist point of view is to treat as real exactly those structures preserved by the maps. These are encoded in $\mathbb{N}$.

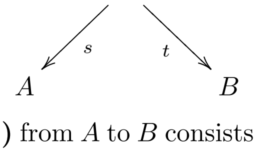
While these examples show in principle how categorification is an explanation of a formal structure (arithmetic) in terms of something more concrete (sets), the theory of "entities which can be counted" is not very sophisticated. To account for quantum mechanics, more is needed, including a decategorification operation which gives more than just bare sets.

2.2. **Spans and Histories.** One important physical principle here is that the right idea of an arrow between two systems is not a *function*, but a *process*. For the moment, we consider this principle without being concerned with categorification, and just consider how it can produce linear structures like Hilbert spaces.

A physical system is formally represented by some sort of space of configurations, which is a set, perhaps with some structure (for instance, a manifold, though this raises extra issues not pertinent now). As a system has a set of configurations, so a process has a set of histories, which link starting states to ending states.

This is the context of Langrangian mechanics, which classically is concerned with selecting a particular history from a space of possibilities as physical. In quantum theory, the "sum over histories", or path integral, means that all histories in a space of possibilities contribute to the transition from a configuration of a system at time t_1 to a configuration at time t_2.

This can be abstracted into a category **Span(Set)**, whose objects are sets (for clarity, suppose they are finite), but which has different arrows from **FinSet**. These are *spans*, diagrams like this:



That is, an arrow in **Span(Set)** from A to B consists of an object H and two arrows s and t from Set. The relevant interpretation is that A and B describe two different systems by collecting their possible configurations. Then H is a set of *histories* characterizing some process which takes A-configurations to B-configurations. The functions s and t pick the starting and terminating configuration for each history $h \in H$, which are configurations of the A-system and B-system respectively. These are unique for each history.

Spans H_1 from A to B and H_2 from B to C can be composed to give a process whose set of histories from $a \in A$ to $c \in C$ is a sum over all intermediate states $b \in B$ of composite histories (h_1, h_2) passing through b.

By **Rel**, we mean the category whose objects are (finite) sets, and whose arrows R from A to B are relations, namely subsets $R \subset A \times B$, consisting of those pairs (a, b) which are related by R. The objects of **Rel** are, internally, exactly the same as the objects of Set, but it has more arrows (since every function is a relation, but not the reverse). A general span H can be described as a "witnessed" relation: the histories in $(s, t)^{-1}(a, b)$ (i.e. with source a and target b) are "witnesses" to the fact that a and b are related (by those histories in H). **Rel** is therefore a simpler cousin of **Span(Set)**, which ignores different such witnesses.

There is a map $U : \mathbf{Span(Set)} \to \mathbf{Rel}$, which takes a span of sets to the relation which indicates whether the set $(s, t)^{-1}(a, b)$ is empty or not. This map gets along with composition. The relation $U(H, s, t)$ says "a and b are related if it is possible to evolve from configuration a to configuration b by a history h of process H".

There is a way to get Hilbert spaces and linear maps from spans of sets, namely the linearization map we mentioned above:

$$L : \mathbf{Span(Set)} \to \mathbf{Hilb} \tag{1}$$

For an object A, it gives $L(A)$, the space of functions on A, with an obvious inner product (from the distinguished basis of delta-functions δ_a for a in A).

A linear map is found by by "pulling and pushing" functions through the span. The image of the function $f \in L(A)$ is the function which at $b \in B$, is the sum of the $f(s(h))$ over the contributions of all histories h ending at b. This amounts to multiplication by a matrix whose (a, b) component is:

$$L(H)_{a,b} = |(s, t)^{-1}(a, b)|$$

This is already a surprisingly powerful setup, as discussed below, but it can be refined even further. The *degroupoidification* process, extends this to groupoids, where we incorporate the second of our two physical motivations.

2.3. **Symmetry and Groupoids.** The second basic physical notion which is important here is symmetry, which plays a crucial role throughout physics. Hermann Weyl wrote [12]: "As far as I can see, all a-priori statements in physics are based on symmetry". Noether's Theorem, for instance, identifies how symmetry actions give rise to quantities which are conserved (and therefore have special physical meaning). Other examples of its special role include: Special and General Relativity (with the Lorentz and diffeomorphism groups as symmetries); gauge symmetry in quantum field theory; and the symmetric tensor products involved in constructing a Fock space of bosons. In each case, the symmetries of formal representations of configurations play a role in specifying those differences which are *physical*.

Groupoids, which can be thought of as representing "local symmetries" of a set [11], are categories where all arrows can be inverted. They generalize the idea of a group, since a group can be thought of as a groupoid with just one object: then the group elements are the arrows of the groupoid. On the other hand, groupoids generalize sets: a set S can be seen as the objects of a groupoid whose only arrows are identities for each element (there are no nontrivial symmetries between elements). There is a transformation groupoid $S//G$ associated to a set S equipped with an action of the group G. Its objects are elements of S, and there are arrows from s to s' corresponding to each g where $g(s) = s'$.

Whatever extra structure they have as spaces, configuration spaces can be extended to have a groupoid structure by including symmetries of configurations as arrows. Each such arrow tells us that two states are "the same" in the sense of being related by a symmetry of the theory, and are *physically* indistinguishable, at least internally. For example, states which can be transformed into each other by a rotation, say, are formally but not physically different.

Then the concept of a map between spaces of some kind can be generalized to that of a functor between groupoids. This includes a map between spaces of objects, and a compatible map between spaces of arrows which respects sources, targets, and composition. That is, the right notion of a map between configuration groupoids is one which respects all symmetry relations.

2.4. **Degroupoidification.** Combining our two physical principles, histories themselves should have symmetries, and the space of histories relating two systems should be represented as a span of groupoids, just as for sets. This is "doing physics" in **Span(Gpd)**. This category has groupoids for its objects, and spans of groupoids (with the right kind of maps just described) for arrows. It is slightly different from **Span(Set)**. The source and target maps must now map symmetries of a history h to symmetries of its source and target states. Composing spans now gives a groupoid of histories whose objects are not just matching pairs of histories, but also include a symmetry which "glues" the source of one to the target of the other. This recognizes that the intermediate objects may be the same *in more than one way.*

Groupoidification is a kind of explanation of particular Hilbert spaces and maps between them, in terms of groupoids and spans of groupoids. Such an explanation is recognizable because the original structure appears under "degroupoidification", the way $\mathbb{N}$ appeared when we decategorified **FinSet**. Degroupoidification is the analog of linearization of sets and spans:

$$D : \mathbf{Span(Gpd)} \to \mathbf{Hilb} \tag{2}$$

Given a groupoid, a Hilbert space can be built from it by taking the the space of *invariant* functions on objects. Transporting such a function through a span again amounts to a "pull-push" process expressible as multiplication by a matrix. As before, we say:

$$D(H)_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

The components are labelled by the sets $\underline{A}$ and $\underline{B}$ of isomorphism classes of objects, such as $[a] \in \underline{A}$.

But now, the space of histories relating a and b is a groupoid, and this expression counts it with "groupoid cardinality", $|G| = \sum_{[x]\in\underline{G}} \frac{1}{|\mathrm{Aut}(x)|}$. This adds up the isomorphism classes in G, but counts each with a weight, so that objects with larger symmetry group $Aut(x)$ count for less. This counting gets along with group actions, in that, when a group G acts on a set S, we get $|S/\!\!/G| = \frac{|S|}{|G|}$.

This process therefore casts the "sum over histories" of Lagrangian quantum mechanics as a decategorification, the way "counting" was in our categorification of $\mathbb{N}$. Finding particular composite histories which give a linear map is then an "explanation" for matrix multiplication.

A groupoidification is a structure in **Span(Gpd)** whose image under D is the chosen one in **Hilb**. It will introduce new information, which we want to construe as explanatory.

One physical example uses $\mathbf{FinSet_0}$, the groupoid whose objects are finite sets and whose arrows are bijections [9] . This is a groupoidification of the *Fock space* for the quantum harmonic oscillator, and the isomorphism classes of objects are the classical configurations of the oscillator: nonnegative integer "energy levels". Acting on it is a groupoidification of the Weyl algebra of operators on Fock space, whose elements are spans of groupoids, composed of some basic building-blocks. The objects of these groupoids can be identified as Feynman diagrams. Applying D, we are performing a sum over histories of the elements of this ensemble of diagrams, weighted by symmetry. (Some further details allow for complex amplitudes by allowing objects to carry phase angles).

This example of groupoidification is a candidate for an explanation of the quantum harmonic oscillator, which directly gives it concrete meaning. Our point is not necessarily that this explanation gives a correct model of underlying content of the objects which represent systems. Indeed, as the categorification of arithmetic shows, there may be different explanations of this sort, perhaps related by structure-preserving maps.

In accord with the structural realist view, we expect that the role of Feynman diagrams in calculating amplitudes for the oscillator should be preserved by a map from this stucture, just as the product structure in *Set* is preserved by monoidal functors between different explanations.

3. Interpretation

Suppose we take seriously the explanation of quantum phenomena that groupoidification suggests. We want to consider now what sort of physical phenomena are "explained" by groupoidification, and what it says about structures (hypothetically) underling the Hilbert space formalism of quantum mechanics. We start by considering the categorification of a quantum state.

3.1. States and Costates.

A **state** for an object of a category with a monoidal operation is an arrow out of the unit object, and a **costate** is a arrow into the unit. This idea (adapted from Abramsky and Coecke [1]) captures the view that a state is a "generalized point" in a configuration space.

In *Hilb*, the category of Hilbert spaces, with $\otimes$ as monoidal product, a state in H is therefore just a map from $\mathbb{C}$ - which is determined exactly by a vector $v \in V$. A costate is a linear functional on H, determined by $w \in H$, sending v to $\langle v|w\rangle$. (In Dirac's terms, a state is a "bra", and a costate is a "ket".) Following the quantum-mechanical interpretation, a state can also be understood as a "preparation process", and a costate as a "measurement procedure".

Now, in **Set**, a "state" for a set S is a map from $\mathbf{1}$, which is given exactly by an element of S. There is only one costate (the unique map to $\mathbf{1}$). Similarly, a state in *Gpd* is an object, and there is only one costate.

The concept makes more sense in **Span(Set)** (and likewise in **Span(Gpd)**. A span from $\mathbf{1}$ amounts to a map $\Psi : X \to S$, and we think of X as an "ensemble" of possible histories of the preparation process the state describes. The map Ψ shows which configuration in S is the endpoint of each history. The unique element (or object) of $\mathbf{1}$ indicates that the possible starting points are undifferentiated.

A general state is thus an *ensemble*, each element of which has some underlying pure state. The arrows in these groupoids describe the symmetries of both the elements of the ensemble X, and the pure states in S. Degroupoidification counts these ensembles - using groupoid cardinality - and gives a vector in the vector space $D(S)$. Our claim is that this constitutes a possible *explanation* of the meaning of a state vector in a Hilbert space. It is analogous to a thermodynamic description: the "true" (i.e. groupoidified) state is an ensemble of micro-configurations, each of which determines a macro-configuration, or element of S.

Composing a state with a costate gives a groupoid whose total size, as measured by D, gives the amplitude for the measurement, given that a state has been set up by the measurement process.

So spans of sets is enough to abstractly account for phenomena such as entanglement. Spans of groupoids, however, seem to be necessary to give a categorification with the potential to explain even the simplest of real quantum systems, the harmonic oscillator.

3.2. **Entanglement.** We have remarked that the category **Span(Set)**, though not a categorification of vector spaces, already has some powerful explanatory features. These are inherited by groupoidification, and are a key part of how it accounts for the phenomenon of entanglement.

The earlier example of categorifying arithmetic relied on a few properties of the category *Set*, of sets and functions: namely, that for any two sets, there is a Cartesian product and a disjoint union. These can be defined abstractly in terms of the category: that is, without directly referring to elements, but only to maps. For example, a product $A \times B$ is an object equipped with projection functions into A and B satisfying a universal property (roughly any other object X with functions f and g into A and B is automatically equipped with a function (f, g) into $A \times B$).

It has been shown in a toy model introduced by Spekkens [10, 3] that **Rel** already supports toy models in which quantum phenomena, in particular entanglement, appear. Spekkens' model is intended to support an epistemic understanding of quantum states. This is that properties reflected by states are not about the internal structure of the system, which is to say the precise nature of the object A. Rather, they pertain to relations to other systems, described in terms of the arrows incident to A, which are associated with physical processes.

Entanglement phenomena occur in **Span(Set)** and **Span(Gpd)**, in much the same way, and our view of groupoidification suggests this as an explanation of these phenomena in quantum mechanics in Hilbert spaces. Considering the case of **Span(Set)** makes this clearer. In *Set*, a state for a set A is an element, and a state for $A \times B$ (the set of pairs of elements) is therefore determined exactly by a pair of states, for A and B. But in **Span(Set)**, and thus in **Rel**, although the objects are the same, and so the "internal description" of the system is the same, the arrows are different.

Now, a span into the product $A \times B$ amounts to an ensemble $\Phi : W \to A \times B$. This is the same as a pair of ensembles $W_A \to A$ and $W_B \to B$ separately only when W itself is a product $W = W_A \times W_B$, and Φ is defined appropriately using the projection maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$. This is typically not the case, so most states of a composite system exhibit entanglement effects.

In categorical terms, this is an expression of the fact that $\times$ is a *categorical product* in the category *Set*, but only a *monoidal product* in **Span(Set)**. This explanation of entanglement, then, is consistent with the view that, while systems may internally have the same form as in a classical system, properties relating to quantum states are epistemic. Moreover, this means that they relate to concrete processes of interaction with the system: in this case, the view of a *state as preparation procedure* of Coecke's monoidal category framework.

In light of our understanding of states as morphisms in a category of spans, which represent processes and which determine relations, we see this accords with Esfeld's [7] account of entanglement in terms of a metaphysics in which properties (costates) are inherently relational.

3.3. **Interference Phenomena.** The account of entanglement above relies only on the first of our two physical principles. We have argued that quantum states are then epistemic, because they refer to arrows describing processes, rather than internal structure of the objects. Still, the groupoid of configurations which describes a system does have internal structure.

The degroupoidification map D is not exactly analogous to L, linearization of spans of sets. Counting a set determines it up to isomorphism, but a groupoid's cardinality does not determine it, even up to equivalence. Two groupoids with cardinality 1 include that with one object and only the identity arrow, and one with two distinct objects, each with both identity and non-identity self-symmetries: $1 = \frac{1}{2} + \frac{1}{2}$, but these express cardinalities of different groupoids. So D *does not faithfully reflect* all the information in **Span(Gpd)**: it only recognizes states up to a sum over histories, counting by groupoid cardinality, and so loses information about a state.

Thus the Hilbert space formalism will sometimes represent different (groupoidified) quantum states by the same vector in Hilbert space. This initially surprising fact just reflects that a groupoidified state is more than the Hilbert space vector precisely in equipping it with an explanation. Two such states may arrive at the same amplitudes for various underlying configurations, but for different reasons - that is, via a different ensemble of "micro-configurations", and the groupoidified state reflects this.

Indeed, this shows that even the simplified setup we have described here supports a phenomenon which appears in more recognizable form in the example of the harmonic oscillator.

There, groupoids can also carry phases, which appear as weights on objects in groupoid cardinalities. A nonempty groupoid can then have cardinality equal to zero, which accounts for the possibility of destructive interference. Thus, a groupoidified state with multiple histories ending at a given configuration yields the same amplitude as one with no such histories, where the final configuration is unreachable.

However, it is worth noting the case of zero amplitude is the only case where the nonuniqueness of the groupoidification requires these phases. With or without them, the underlying structural cause is that the decategorification can produce the same a state *vector* in a Hilbert space for potentially very different groupoids of histories, and that it is precisely the *history* of a configuration which makes the difference.

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