Categorification and Groupoidification of the Heisenberg Algebra

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Abstract

These lectures, prepared for Higher Structures in China III, held in Changchun, Aug 2012, describe a relationship between two forms of categorification of algebras by giving a combinatorial model for Khovanov's categorification of the Heisenberg algebra in a 2-category of spans of groupoids. This is joint work with Jamie Vicary.

The goal here is to describe two notions of "categorifying an algebra", and see how they are related in one special example - that of the Heisenberg algebra.

The most important difference is a higher-categorical analog of the difference between a **presentation** and a **representation** of an algebra. The model we find via Baez-Dolan-Trimble "groupoidification" is a concrete representation of the abstract categorification of Khovanov.

There are two lectures, which will cover the following:

Lecture 1

- Categorification and Decategorification
- The Heisenberg Algebra and the Fock Space Representation
- Groupoidification and the Heisenberg Algebra

Lecture 2

- Diagrammatic Categorification of the Heisenberg Algebra
- Biadjointness and "process movies"
- The Combinatorial Representation

1 First Lecture (Aug 13, 2012)

1.1 Categorification and Decategorification

A **decategorification** operation is one which takes a category with structure to a set with related structure.

Example 1.1. The *cardinality* operation: takes $S \in \mathbf{Sets}$ to its isomorphism class, which is labelled by the number $|S| \in \mathbb{N}$. So $\mathbb{N}$ is a decategorification of **Sets**. **Sets** also has structure respected by $|\cdot|$:

Category	Set
Sets	$\mathbb{N}$
Monoidal ($\otimes = \times$)	Monoid ($\cdot$)
Symmetry	Commutative
Coproducts ($\coprod = \sqcup$)	Commutative Monoid ($+$)

The *Grothendieck Ring* of **Sets** is $\mathbb{Z}$. In general, the Grothendieck ring takes *stable equivalence classes*, where $A \simeq B$ whenever there is some C so that $A \coprod C \cong B \coprod C$. Then it takes the completion of the commutative semiring, as we just found, to a ring.

The decategorification $|\cdot|$ takes objects in the category **Sets** (up to isomorphism) to elements of the set $\mathbb{N}$. Not every kind of decategorification does this: below, we describe *degroupoidification*, in which the elements of the set do not correspond directly to particular objects.

Given a decategorification operation D, a **categorification** of a structured set S, relative to D, is any categorical structure $\mathcal{S}$ such that $D(\mathcal{S}) = S$. Our goal here is to show a relationship between two apparently different kinds of categorification.

1.2 The Heisenberg Algebra and the Fock Space Representation

The Heisenberg Algebra is the free complex algebra on generators $\mathbf{a}^\dagger$ and $\mathbf{a}$, satisfying the commutation relation

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1. \tag{1}$$

This is an abstract presentation of the algebra, but for practical purposes, it is often more important to know that the Heisenberg Algebra has an essentially unique faithful representation on a Hilbert space. This algebra and the Hilbert space form the description of the quantum harmonic oscillator in physics.

One description treats *Fock Space* (the underlying space of the representation) as the Hilbert space $\mathbb{C}[\![z_1, \dots, z_n]\!] = \{a_0 + a_1 z + a_2 z^2 + \dots\}$ of power series in z. It is a separable Hilbert space with a basis consisting of the z^n. It is given the inner product where these are orthogonal, but $\langle z^n, z^n \rangle = n!$.

The *creation* and *annihilation* operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ act on Fock space by:

$$\mathbf{a} = \partial_z \tag{2}$$

and

$$\mathbf{a}^\dagger = z \tag{3}$$

These are the differentiation and multiplication operators. The canonical commutation relation $[\mathbf{a}, \mathbf{a}^\dagger] = Id$ says:

$$\partial_z(zp(z)) - z\partial_z(p(z)) = p(z) \tag{4}$$

which is just the Leibniz rule for derivatives.

Intuitively, this is a physical system in which most important observable quantity is the number of particles. It is a space of linear combinations of states, where the vector z^n represents a state in which there are n particles in the system. The operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ act on any basis state by either adding or removing a particle.

It is important that $\mathbf{a}$ and $\mathbf{a}^\dagger$ are *adjoints* with respect to the inner product on Fock space:

$$\langle \mathbf{a}p(z), q(z) \rangle = \langle p(z), \mathbf{a}^\dagger q(z) \rangle \tag{5}$$

Intuitively, this means they are "time reversed" versions of each other.

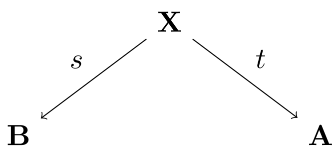
1.3 Groupoidification of the Heisenberg Algebra

"Groupoidification" is a kind of categorification of linear structures - namely, those living in a category **Vect** of (complex) vector spaces. It is a categorification in the sense mentioned above: it is a non-rigorous process, but we have a way to decide when we have a groupoidification of a given structure. To explain this, we first need to know what kinds of categories will stand in for vector spaces, and what kind of morphisms will stand in for linear maps in a groupoidification.

What follows is a superficial account, but see, for instance, [1] or [2] for more details.

Definition 1.2 (Part 1). *The monoidal (1-)category* **Span**(**Gpd**) *has:*

- ***Objects*** *Groupoids (categories whose morphisms are all invertible) with some nice property ("tame" - see [2] or [3] for more details).*
- ***Morphisms*** *Spans of groupoids:*



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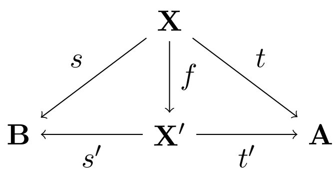
(which we occasionally write $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$*), taken up to isomorphism of spans, namely isomorphisms* f *forming commuting diagrams like:*



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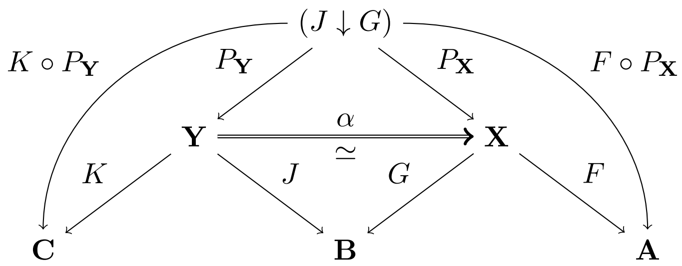
- *Spans are composed by weak pullback as below.*

(These pictures are read right-to-left.)

The composition works as follows. Given spans of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ and $\mathbf{C} \xleftarrow{K} \mathbf{Y} \xrightarrow{J} \mathbf{B}$, we can compose them by constructing a *weak pullback* groupoid $(J \downarrow G)$:



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This weak pullback is defined by a universal property, but a standard construction for $(J \downarrow G)$ is:

- **Objects** are triples $\big(x \in \mathrm{Ob}(\mathbf{X}), y \in \mathrm{Ob}(\mathbf{Y}), G(x) \xrightarrow{f} J(y)\big)$.

- **Morphisms** $(x_1, y_1, f_1) \to (x_2, y_2, f_2)$ are pairs of morphisms $x_1 \xrightarrow{a} x_2$ and $y_1 \xrightarrow{b} y_2$ satisfying the following commuting diagram:

$$\begin{array}{ccc} G(x_1) & \xrightarrow{f_1} & J(y_1) \\ {\scriptstyle G(a)}\downarrow & & \downarrow{\scriptstyle J(b)} \\ G(x_2) & \xrightarrow[f_2]{} & J(y_2) \end{array} \tag{9}$$

This construction is essentially a "weak" form of the fibred product, where instead of taking pairs (x, y) whose images in $\mathbf{B}$ agree, we choose a specific isomorphism between them.

Now, with $\mathbf{Span}(\mathbf{Gpd})$ defined, there are several equivalent ways to define degroupoidification, but one is as follows:

Definition 1.3 (Baez-Dolan)**.** *The **degroupoidification functor** acts on*

$$D : (\mathbf{Span}(\mathbf{Gpd})) \to \mathbf{Hilb} \tag{10}$$

assigns to a groupoid G

$$D(G) = \mathbb{C}^{\underline{G}} \tag{11}$$

the space of equivariant functions on the objects of the groupoid (or functions on isomorphism classes). This is given an inner product where

$$\langle \delta_a, \delta_b \rangle = \delta_{a,b} \# Aut(a) \tag{12}$$

To a span (X, s, t), D assigns the linear map

$$t_* \circ s^* : D(A) \to D(B) \tag{13}$$

where

$$s^* : \mathbb{C}^{\underline{A}} \to \mathbb{C}^{\underline{X}} \tag{14}$$

acts by composition with s, and t_* is the $\langle \cdot, \cdot \rangle$-adjoint of t^*.

The adjoint of t^* amounts to a "push-forward" map, which takes a function on $\underline{X}$ to a function on $\underline{B}$ by:

$$(t_* f)([b]) = \sum_{[x] \in \underline{t^{frm[o]--}}} ([b]) \frac{\# Aut(x)}{\# Aut(b)} f([x]) \tag{15}$$

This is a "weighted sum over histories".

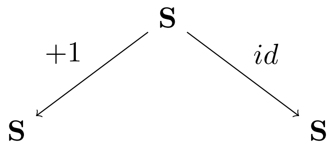
There is a groupoidification of the Fock space representation of the Heisenberg algebra. It makes the following assignments.

Fock Space: the groupoid $\mathbf{S}$, whose objects are *finite Sets*, and whose morphisms are *bijections*.

The **raising** and **lowering operators** respectively, the spans:

$$\begin{array}{ccccc} & & \mathbf{S} & & \\ & {\scriptstyle id}\swarrow & & \searrow{\scriptstyle +1} & \\ \mathbf{S} & & & & \mathbf{S} \end{array} \tag{16}$$

and



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That is, (if we identify $D(\mathbf{S})$ with $\mathbb{C}[\![z_1, \ldots, z_n]\!]$) then $D(A) \cong \mathbf{a}$ and $D(A^\dagger) \cong \mathbf{a}^\dagger$.

Note that these two spans look just the same, except for the sense of which arrow is the "source map" and which is the "target map". We can say that they are "duals". It is an obvious consequence of the definition we picked for groupoidification that these dual spans give adjoint linear maps in the inner product on $D(\mathbf{S})$.

This groupoidification gives a *combinatorial interpretation* of these operators, and therefore of their composites. They are literally "rasing" and "lowering" the number of elements in a set.

It is shown in [6] how this combinatorics relates to the fact that certain interesting algebra elements, and inner products, can be calculated using *Feynman diagrams*.

2 Second Lecture (Aug 15, 2012)

The point of this talk is to show that our groupoidification is (part of) a concrete model of Khovanov's categorification of the Heisenberg algebra [4]. This is a diagram calculus defining a monoidal category $\mathbf{H'}$ with a zero object and biproducts, in terms of generators and relations. The Grothendieck ring of this category is isomorphic to (the integral part of) the Heisenberg algebra.

That is, Khovanov's $\mathbf{H'}$ gives a presentation of what a categorified Heisenberg algebra must be, and our groupoidification gives a particular representation. For this to make sense, we will need to describe the natural 2-morphisms of **Span**(**Gpd**). This makes it possible to represent $\mathbf{H'}$ in the category of endomorphisms of an object in **Span**(**Gpd**), namely **S**. This is analogous to a representation of an algebra in the endomorphisms of an object in **Vect** - i.e. as operators on a vector space.

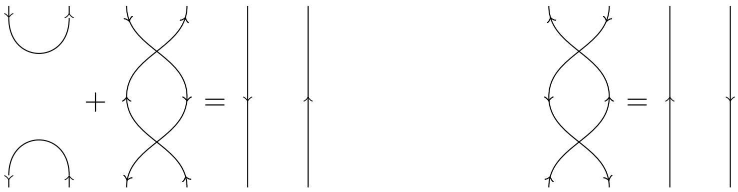
2.1 Diagrammatic Categorification of the Heisenberg Algebra

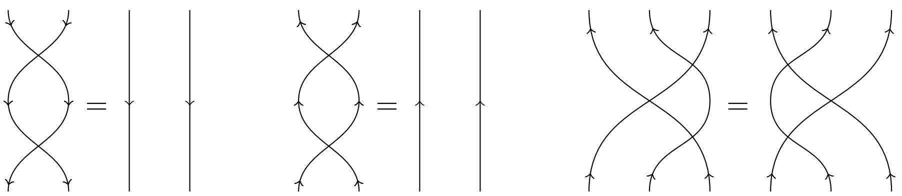
Just as we started by defining a presentation of the Heisenberg algebra, we now start with a presentation of Khovanov's $\mathbf{H'}$. Its presentation has two generating objects Q_+ and Q_-, written graphically as upwards- and downwards-pointing strands:

The monoidal product is represented by horizontal juxtaposition of diagrams.

The morphisms are written as (linear combinations of) equivalence classes of planar diagrams. The diagrams are taken up to planar isotopy and a few other relations. The generating morphisms are written as cups, caps, and crossings with various orientations:

Now, apart from isotopy, the relations are imposed on the generators are:





(And a similar Reidemeister-type relation on three downward-pointing strands.) The last three imply that the crossings of strands with the same orientation just act like generators of the symmetric group.

The first two equations make up injection and projection maps for the biproduct:

$$Q_- \otimes Q_+ \simeq Q_+ \otimes Q_- \oplus I \tag{18}$$

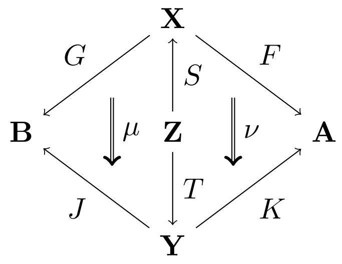
where I is the monoidal unit object. This is a categorification of the Heisenberg algebra relation (1).

2.2 Biadjointness and "Process Movies"

We said we want to find the groupoidified Heisenberg algebra as a model of the diagrammatic categorification, acting by *endomorphisms* on an object in **Span**(**Gpd**). Since the raising and lowering operators are objects in the categorification, and spans in **Span**(**Gpd**), we have a question. There are morphisms between such objects, so we need morphisms between spans in **Span**(**Gpd**).

The simplest kind of morphism is a *span map*, namely diagrams like (7). In **Gpd**, which has 2-morphisms already, we only need to ask that this commutes up to a specified 2-morphism.

For various reasons, it is sometimes good to use *spans of* span maps:



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We can compose these just in the same way we usually compose spans, by weak pullback. These are the 2-morphisms of the full 2-category **Span**(**Gpd**). The 1-category we used in Lecture 1 is really the *homotopy category* of this: the 1-morphisms there are really *isomorphism classes* of 1-morphisms in the full 2-category, and all 2-morphisms are otherwise ignored.

Intuitively, the idea is that a span of groupoids describes a groupoid of possible *histories* for a *process* which relate starting and ending configurations, which are the objects of the source and target groupoids. These 2-morphisms should now be thought of as "process movies": processes by which we relate one collection of histories to another. In our combinatorial interpretation later, we will see some examples.

The reason this 2-category turns out to be useful because it contains every object and morphism of **Gpd**, and now every morphism has a two-sided adjoint: for every span, its converse is both a left and a right adjoint. This situation is called an *ambidextrous adjunction*, or just *ambijunction* for short.

(Note: if we had used span maps for 2-morphisms, this wouldn't have worked!)

This means that there are certain 2-morphisms, the units and counits:

$$\eta_L : Id_A \to F \circ F^\dagger \tag{20}$$

$$\eta_R : Id_B \to F^\dagger \circ F \tag{21}$$

$$\epsilon_L : F^\dagger \circ F \to Id_B \tag{22}$$

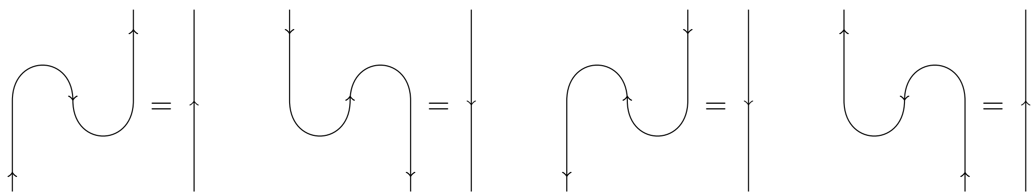
$$\epsilon_R : F \circ F^\dagger \to Id_A \tag{23}$$

These satisfy the usual adjunction properties:

$$(Id \circ \eta_L) \cdot (\epsilon_L \circ Id) = Id \tag{24}$$

$$(\eta_R \circ Id) \cdot (Id \circ \epsilon_R) = Id \tag{25}$$

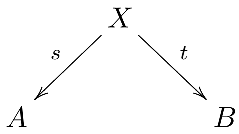
which correspond exactly to the "zig-zag" rules:



These hold in the diagram category $\mathbf{H}'$ automatically because we consider diagrams up to planar isotopy - and all these equations look like "yanking" a strand. So they must hold anywhere $\mathbf{H}'$ is to be represented: any object in $\mathbf{H}'$ must be taken to an endomorphism which has a two-sided adjoint.

Now, this always holds in $Span(Gpd)$, if we know what its 2-morphisms should be!

The span $F : A \to B$ given as



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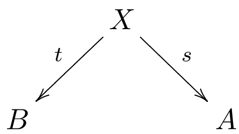
has ambiadjoint $F^\dagger : B \to A$ found by reversing orientation:



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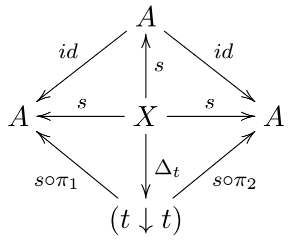
We have $\eta_L = \epsilon_R^{co}$:



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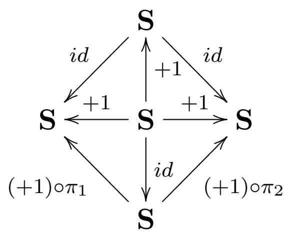
- $(t \downarrow t)$ is the comma category whose objects are (x, f, x') with $f : t(x) \to t(x')$, and whose morphisms are commuting squares
- $\Delta_t : X \to (t \downarrow t)$ takes objects $x \mapsto (x, id_{t(x)}, x)$ and morphisms $g \mapsto (g, g)$

And similarly for $\eta_R = \epsilon_L^{co}$.

Take the case where $F = A$, the groupoidified annihilation operator. Then the left unit $\eta_L : Id_{\mathbf{S}} \Rightarrow A \circ A^\dagger$ is (equivalent to):



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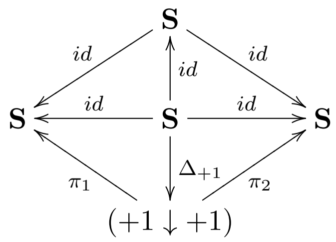
And the right unit $\eta_R : Id_{\mathbf{S}} \Rightarrow A^\dagger \circ A$ is:



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The groupoid $(+1 \downarrow +1)$ can be described up to equivalence as having:

- **Objects**: (n, ϕ, n), where $\phi \in S_{n+1}$

- **Morphisms**: $(\pi_1, \pi_2) \in S_n^2$ such that $\phi' \circ \pi_1 = \pi_2 \circ \phi$

We can understand η and its converse combinatorially in the following way.

- $\mathrm{id}_{\mathbf{S}} \xrightarrow{\eta} A^\dagger \circ A$. This relates the identity history to the history where an element is removed, and then added. This is impossible on the zero-element set, and so in that case η relates the identity history to nothing.

- $A^\dagger \circ A \xrightarrow{\eta^\dagger} \mathrm{id}_{\mathbf{S}}$. This relates any history to the identity history.

2.3 The Combinatorial Interpretation

An important interpretation of groupoidification comes from physics: groupoids represent *physical symmetries*, and spans represent *spaces of histories*. The idea is that configuration spaces of physical systems can be represented as groupoids in a way that usefully encodes their symmetries, and that spans of groupoids encode ways in which these states and their symmetries can be transformed via physical processes.

In our example of the harmonic oscillator, these configurations are the integer-valued energy eigenstates - the integers count the number of elements in any set in $\mathbf{S}$.

A span represents a space of *histories*, with its source and target maps picking out the starting and ending configurations. Furthermore, these are not just set-maps of the objects (histories and configurations), but functors, which also describe how symmetries of histories act on starting and ending configurations.

This interpretation lets us give combinatorial accounts of the relations imposed in the definition of $\mathbf{H'}$. For instance, we can say in summary that the canonical commutation relations mean: *"add-then-remove" has one more possibility than "remove-then-add"*.

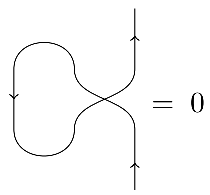
For instance, in the first of the imposed relations:

- The RHS shows the identity on $A^\dagger \circ A$

- First term on LHS swaps the order to give $A \circ A^\dagger$ (selects the case "remove a different element from that added"

- Second term selects the case "remove the same element added" (otherwise the counit is zero)

Similarly, there is an interpretation of the relation:



This is that a certain sequence of changing processes cannot be done:

- Add new element x into a set

- *(insert add-remove pair)*

- Add new element x, then y, then remove y

- *(swap adding x and y)*

- Add y, then x, then remove y

- *(cancel add-x-remove-y pair: IMPOSSIBLE)*
- Add y

This says that "x is different from y", or "elements of sets are distinguishable".

2.4 Other Representations

If we want a representation of $\mathbf{H}'$ on a category *by functors and natural transformations*, there are some natural starting points once we have the concrete groupoidification:

- As in [6], the groupoidified Heisenberg algebra has a natural action on the category of all *stuff types* - a very wide class of combinatorial constructs which have underlying finite sets.
- As we describe in [5], there is a "linearization" process Λ, which replaces the groupoid $\mathbf{S}$ with its representation category, and gives a functors for spans by a "pull-push" process. So there is a representation on $Rep(\mathbf{S})$. This is equivalent to the representation in a bimodule category in [4].

References

[1] John Baez and James Dolan. From finite sets to Feynman diagrams. In B. Engquist and W W. Schmid, editors, *Mathematics Unlimited - 2001 And Beyond*. Springer Verlag, 2001. `arxiv:math.QA/0004133`.

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[6] Jeffrey C. Morton. Categorified algebra and quantum mechanics. *Theory and Applications of Categories*, 16:785–854, 2006. `http://www.tac.mta.ca/tac/volumes/16/29/16-29abs.html`.