Extended TQFT from (Higher) Gauge Theories

Jeffrey C. Morton

Instituto Superior Técnico

Higher Gauge Theory, TQFT, Quantum Gravity
Lisbon, Portugal
Feb 2011

Definition

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Vect}$$

where $\mathbf{nCob_2}$ has

- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)
- **2-Morphisms**: n-dimensional cobordisms with corners

We'll construct an ETQFT by factoring through a 2-category $Span(\mathbf{Gpd})$, then applying some universal process.

"Physical" applications of groupoids arise mostly from *action groupoid* $S /\!\!/ G$ associated to a G-action on S, where S is a space of configurations. (Secretly the groupoid is a *stack*.)

Example

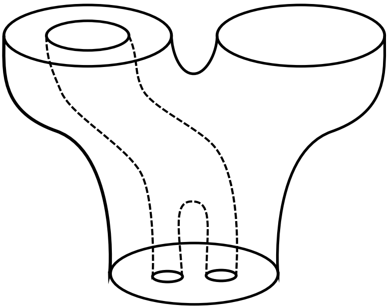
Moduli space for *gauge theory*, for (finite) gauge group G. Given M, the groupoid $\mathcal{A}_0(M, G) = hom(\pi_1(M), G) /\!\!/ G$ has:

- **Objects**: Flat connections on M
- **Morphisms** Gauge transformations

Goal: Using the induced 2-functor $\mathcal{A}_0(-, G) : \mathbf{nCob_2} \rightarrow Span_2(\mathbf{Gpd})$, we get an ETQFT $Z_G = \Lambda \circ \mathcal{A}_0(-, G)$.

Problem: What?

This relies on the fact that cobordisms in $\mathbf{nCob_2}$ actually live in $Span^2(\mathbf{ManCorn})$, as double cospans (here $n = 3$):

$\mathbf{nCob_2}$	$Span^2(ManCorn)$
	$\begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A\coprod D) & \xleftarrow{i'_A\otimes i_D} & S^1\coprod S^1 \\ \downarrow i_1 & & \downarrow \iota_1 & & \downarrow i_2 \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ \uparrow i_2 & & \uparrow \iota_2 & & \uparrow i_1 \\ S^1\coprod S^1 & \xrightarrow[i_2]{} & Y & \xleftarrow[i_1]{} & S^1 \end{array}$

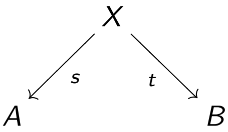
These form a "double bicategory", but it can be coerced into becoming a bicategory since horizontal and vertical morphisms are composable.

Theorem

There is a 2-functor ("2-linearization"):

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

A *span* in a category **C** is a diagram:



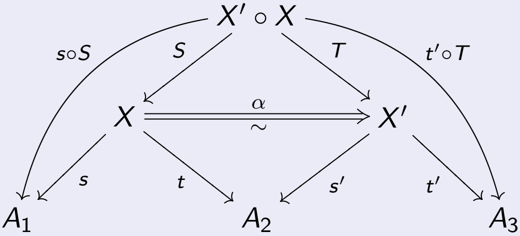
In a span $A \leftarrow X \rightarrow B$, think of X as a space (**C**-object) of *histories*; intuitively s and t pick the starting and terminating *configuration* in spaces A and B. (Only true if **C** is concrete.)

For groupoids, spans also go by "Morita morphisms", etc.

The bicategory $Span_2(\mathbf{Gpd})$ (similar for any 2-category with weak pullbacks) has:

Definition (Part 1)

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- Composition defined by *weak* pullback:

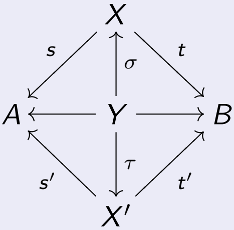


- monoidal structure from the product in **Gpd**, monoidal unit 1

(We could stop here: $Span_1(\mathbf{C})$ is the universal category containing $\mathbf{C}$ with duals for morphisms. But $\mathbf{C} = \mathbf{Gpd}$ is a 2-category).

Definition (Part 2)

The **2-morphisms** of $Span_2(\mathbf{Gpd})$ are (iso. classes of) spans of *span maps*:



(Which possibly commute only up to 2-morphism of **Gpd** - here we ignore this). Composition is by weak pullback taken up to isomorphism.

(Note: In general, $Span_2(\mathbf{C})$ will be the universal 2-category containing $\mathbf{C}$ in which morphisms have ambidextrous adjoints.)

Definition

2Vect is the 2-category of 2-vector spaces, which consists of:

- Objects: $\mathbb{C}$-linear abelian category, generated by simple objects
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence abelian) functor.
- 2-Morphisms: Natural transformations

Finite dimensional 2-vector spaces all look like $\mathbf{Vect}^k$, and 2-linear maps have a matrix representation. (Analogous examples occur for infinite dimensional 2-vector spaces).

Lemma

If $\mathbf{B}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$ *is a KV 2-vector space.*

The generators of $[\mathbf{B}, \mathbf{Vect}]$ are irreducible reps - labeled by $([b], V)$, where $[b] \in \underline{\mathbf{B}}$ and V an irreducible rep of $Aut(b)$.

Theorem

If $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \rightarrow \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \rightarrow \Lambda(\mathbf{X})$$

with $f^* F = F \circ f$ *and (the restricted representation along* f*)*

$$f_* : \Lambda(\mathbf{X}) \rightarrow \Lambda(\mathbf{B})$$

the induced representation of F *along* f*. Furthermore,* f_* *is the two-sided adjoint to* f^**.*

In fact, the adjoint map f_* acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x)\cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a (left) *Kan extension* of the functor F along f.
This is the left adjoint. But there is also a right adjoint (right Kan extension):

$$f_!(F)(b) \cong \bigoplus_{[x]|f(x)\cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

There is the canonical *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \to f_*(F)(b)$$

given by the *exterior trace map* (which uses a modified group average in each factor):

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

Under this identification we get that f^* and f_* are ambidextrous adjoints.

Call the adjunctions in which f_* is left or right adjoint to f^* the *left and right adjunctions* respectively. We want to use the counit for the right adjunction, the evaluation map:

$$\eta_R(G)(x) : v \mapsto \bigoplus_{y|f(y)\cong x} (g \mapsto g(v))$$

and the unit for the left adjunction, which is determined by the action:

$$\epsilon_L(G)(x) : \bigoplus_{[y]|f(y)\cong x} g_y \otimes v \mapsto \sum_{[y]|f(y)\cong x} f(g_y)v$$

These define maps between $F(x)$ and $f_* f^* F(x)$.

(Note: there are canonical inner products around which make these maps *linear* adjoints. We are ignoring them for now.)

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathbf{a}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

$\Lambda(X, s, t)$ can be represented by the matrix with coefficients:

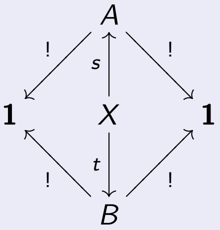
$$\Lambda(X, s, t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$
$$\simeq \bigoplus_{[x]\in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

This is an *intertwiner space* for the groupoid representations. The 2-morphisms give (componentwise) linear maps between intertwiner spaces.

In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

Theorem

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*



where $\mathbf{1}$ *is the (terminal) groupoid with one object and one morphism,* Λ *on 2-morphisms is just the degroupoidification functor D of Baez and Dolan.*

A consequence is that $Z_G = \Lambda \circ \mathcal{A}_G(-)$ gives the *Dijkgraaf-Witten model* when $n = 3$.

Generalization 1: Higher gauge theory - for a 2-group $\mathcal{G}$, define a 3-functor $Z_{\mathcal{G}} : \mathbf{nCob_3} \to \mathbf{3Vect}$.
Sketch:

Definition

Fixing a 2-group $\mathcal{G}$, the contravariant 2-functor

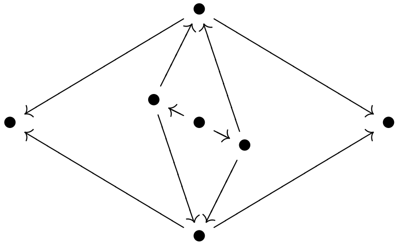
$$\mathcal{A}_0^{(2)} = 2Fun[\Pi_2(-), \mathcal{G}]$$

assigns to a manifold M the 2-groupoid $\mathcal{A}_0^{(2)}(M)$ with:

- Objects: 2-functors ("2-connections")
- Morphisms: natural transformations ("gauge transformations")
- 2-Morphisms: modifications (...)

(and, to smooth functions, the induced maps).

There's an induced map $Span_3(\mathbf{ManCorn}) \to Span_3(\mathbf{2Gpd})$, where $Span_3(-)$ has, as 3-morphisms, equivalence classes of diagrams shaped like:



(1)

Composition is again by weak pullback. (Note that 2-morphisms and 3-morphisms of **2Gpd** can appear in $Span_3(\mathbf{2Gpd})$ by weakening the assumption that this commutes.)

As before, $nCob_3$ lives in $Span^3(\mathbf{ManCorn})$ (cubical, but can be intimidated into being globular if desired).

We would like to define a 3-functor

$$\Lambda^{(2)} : Span_3(\mathbf{2Gpd}) \to \mathbf{3Vect}$$

Then assuming $\Lambda^{(2)}$ is well-defined, we should obtain an extended TQFT 3-functor:

$$Z_{\mathcal{G}} = \Lambda^{(2)} \circ \mathcal{A}_0^{(2)} : \mathbf{nCob_3} \to \mathbf{3Vect}$$

For $\mathcal{X} \in \mathbf{2Gpd}$, we expect to get:

$$\Lambda^{(2)}(\mathcal{X}) = Rep(\mathcal{X}) = 2Fun(\mathcal{X}, \mathbf{2Vect})$$

There should be, for each $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$, an adjoint pair

$$\mathcal{F}^* : Rep(\mathcal{Y}) \to Rep(\mathcal{X})$$

and

$$\mathcal{F}_* : Rep(\mathcal{X}) \to Rep(\mathcal{Y})$$

where the induced representation functor $\mathcal{F}_*$ is given by 2-Kan extension along $\mathcal{F}$.

To prove: It should be biadjoint. Moreover, to get 3-morphisms, the unit and counit ϵ_L, η_R should themselves be biadjoint!

Eventually: One hopes this pattern will repeat with representations of n-groupoids for all n. Must deal with slight trickiness of **nVect**.

Generalization 2: Z_G for G a compact Lie group (uses measured groupoids)

Duplicating the above requires some changes:

- Direct sums become direct integrals - which are not (co)limits
- Push-forward is not just Kan extension of functors
- Topology is nontrivial, so must deal explicitly with sheaves, instead of functors, carrying representations
- Ambi-adjunction requires **Hilb** instead of **Vect** in infinite-dim setting

The construction for Λ can be extended using:

- $Rep(\mathbf{B}) \mapsto$ Category of reps of von Neumann algebra associated to **B**
- 2-linear maps represented by *Hilbert bimodules*
- Natural transformations represented by bimodule maps

This relates to a conjecture (Baez, Baratin, Freidel, Wise) that *infinite-dimensional 2-Hilbert spaces* are representation categories for v.N.-algebras.

2-Vector spaces must be generalized to categories like:

Definition

If (X, μ) is a measurable space **Meas(X)** is the category with:

- Objects: *measurable fields of Hilbert spaces* on $(X, \mathcal{M})$
- Morphisms: *measurable fields of bounded linear maps*

A measurable field of Hilbert spaces on X determines a *measurable sheaf of Hilbert spaces* in $MSh(X, \mu)$ by direct integration.

Theorem (Wendt)

Given a disintegration $f : (X, \mu) \rightarrow (B, \nu)$ *(i.e. morphism in suitable category of measure spaces), there is an adjoint pair:*

$$MSh(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} MSh(Y)$$

Needed: A (groupoid-)equivariant version of this theorem.