# Groupoidification in Physics

Jeffrey C. Morton

Instituto Superior Técnico,
Universidade Técnica da Lisboa

TQFT Club Seminar, IST
Nov 2010

**Motivation**: Categorify a quantum mechanical description of states and processes. We propose to represent:

- configuration spaces of physical systems by **groupoids** (or *stacks*), based on local symmetries
- process relating two systems through time by a **span** of groupoids, including a groupoid of "histories"

This is "doing physics in" the monoidal (2-)category Span(**Gpd**), and relates to more standard formalism by:

- **Degroupoidification**: turns this into physics in **Vect** (or **Hilb**), as usual in quantum mechanics.
- **2-Linearization** gives a more complete equivalence-invariant Λ for Span(**Gpd**). "Physics in **2Hilb**."

Both invariants rely on a **pull-push** process, and some form of **adjointness**.

## Definition

A **groupoid G** (in **Set**) is a category in which all morphisms are invertible. That is, as a category, consists of two sets $G_0$ (of objects) and $G_1$ (of morphisms/arrows) together with structure maps:

$$G_1 \times_{G_0} G_1 \stackrel{\circ}{\to} G_1 \stackrel{s,t}{\to} G_0 \stackrel{i}{\to} G_1 \stackrel{(-)^{-1}}{\to} G_1 \tag{1}$$

which define source, target, identities, partially-defined composition, and inverses, satysifying some properties making a groupoid a "multi-object" generalization of a group.

Morphisms (arrows) of a groupoid can be composed if the source of one arrow is the target of the other. This can be defined where $G_0$ and $G_1$ are sets, topological spaces, manifolds, etc. (Then the maps must be "nice" in a suitable sense in each case.)

## Definition

There is a 2-category **Gpd** with:

- **Objects**: Groupoids (categories whose morphisms are all invertible)
- **Morphisms**: Functors between groupoids
- **2-Morphisms**: Natural transformations between functors

Groupoids provide a good way of thinking about local symmetry. E.g. the transformation groupoid $S /\!\!/ G$ comes from a set $S$ with an action of the group $G$: objects are elements of $S$, morphisms correspond to group elements.

## Example

Some relevant groupoids:

- Any set $S$ can be seen as a groupoid with only identity morphisms
- Any group $G$ is a groupoid with one object
- Given a set $S$ with a group-action $G \times S \to S$ yields a transformation groupoid $S /\!\!/ G$ whose objects are elements of $S$; if $g(s) = s'$ then there is a morphism $g_s : s \to s'$
- Given a differentiable manifold $M$, the **fundamental groupoid** $\Pi_1(M)$ which has objects $x \in M$ and morphisms homotopy classes of paths in $M$.
- Given a differentiable manifold $M$ and Lie group $G$, the groupoid $\mathcal{A}_G(M)$ of **principal $G$-bundles** and bundle maps; and the groupoid $\mathcal{A}_G(M)$ of FLAT $G$-bundles and maps.

Physically, groupoids can describe *configuration spaces* for physical systems. (Many physically realistic cases will also be, e.g. symplectic manifolds, whose points are the objects of the groupoid).

Since groupoids are categories, it is usual to think of them up to *equivalence*. For topological and smooth groupoids, the best version of this is:

### Definition

Two groupoids **G** and $\mathbf{G}'$ are (strongly) **Morita equivalent** if there is a pair of morphisms:

$$\begin{array}{ccc} & \mathbf{X} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathbf{G} & & \mathbf{G}' \end{array} \qquad (2)$$

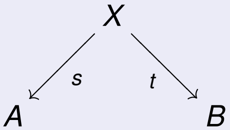
where both $f$ and $g$ are suitably nice maps (otherwise this is a *Morita morphism*). A **stack** is a Morita-equivalence class of groupoids.

- Strong Morita equivalence implies that the categories of representations are equivalent (*weak* Morita equivalence)
- In some cases, they are equivalent (but e.g. not for smooth groupoids)
- coincides with Morita equivalence for $C^\star$ algebras, in the case of groupoid algebras.
- Morita equivalent groupoids are "physically indistinguishable". (E.g. full action groupoid; skeleton, with quotient space of objects).

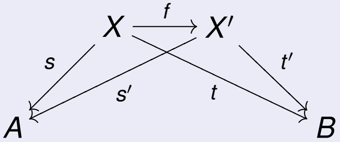
Our proposal is that configuration spaces should be (topological, smooth, etc.) stacks.

## Definition

A **span** in a category **C** is a diagram of the form:




A *span map* $f$ between two spans consists of a compatible map of the central objects:



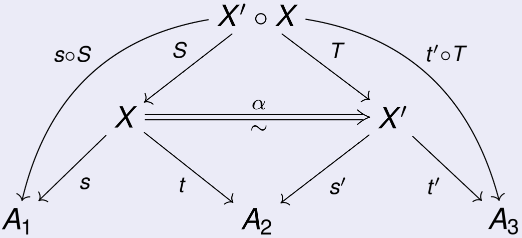

A *cospan* is a span in $\mathbf{C}^{\text{op}}$ (i.e. **C** with arrows reversed).

We'll use $\mathbf{C} = \mathbf{Gpd}$, so $s$ and $t$ are functors (i.e. also map morphisms, representing symmetries).

## Definition

The bicategory $Span_2(\mathbf{Gpd})$ has:

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- Composition defined by *weak* pullback:




(3)

- **2-Morphisms** : isomorphism classes of *spans of* span maps
- monoidal structure from the product in **Gpd**, and duals for morphisms and 2-morphisms.

We can look at this two ways:

- Span **C** is the *universal* 2-category containing **C**, and for which every morphism has a (two-sided) adjoint. The fact that arrows have adjoints means that Span(**C**) is a †-monoidal category (which our representations should preserve).
- Physically, $X$ will represent an object of *histories* leading the system $A$ to the system $B$. Maps $s$ and $t$ pick the starting and terminating *configurations* in $A$ and $B$ for a given history (in the sense internal to **C**).

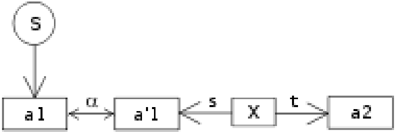
## Definition

A **state** for an object $A$ in a monoidal category is a morphism from the monoidal unit, $\psi : I \to A$.

In **Hilb**, this determines a vector by $\psi : \mathbb{C} \to H$. In Span(**Gpd**), the unit is **1**, the terminal groupoid, so this is determined by:

$$S \stackrel{\Psi}{\to} A$$

where $S$ is a groupoid, "fibred over $A$".
Think of such a state as an **ensemble** over the base groupoid $A$.
Acting on a state for $A_1$ by a span $X : A_1 \to A_2$ produces a state over $A_2$ - an ensemble whose objects include a history:



There is also a category $Span_1(\mathbf{Gpd})$, taking spans only up to isomorphism and neglecting the 2-morphisms, but still composing via weak pullback.
There are two interesting functors for our purposes.
"Degroupoidificatidon" (Baez-Dolan):

$$D : Span_1(\mathbf{Gpd}) \to \mathbf{Hilb}$$

and "2-linearization" (Morton):

$$\Lambda : Span_2(\mathbf{Gpd}) \to \mathbf{2Hilb}$$

### Definition

The **cardinality** of a groupoid **G** is

$$|\mathbf{G}| = \sum_{[g]\in\underline{\mathbf{G}}} \frac{1}{\#\,\mathrm{Aut}(g)}$$

where $\underline{\mathbf{G}}$ is the set of isomorphism classes of objects of **G**. We call a groupoid **tame** if this sum converges.

This has the nice property that it "gets along with quotients":

### Theorem (Baez, Dolan)

*If $S$ is a set with a $G$-action $G \times S \to S$, then*

$$|S /\!/ G| = \frac{\#S}{\#G}$$

*where $\#$ denotes ordinary set-cardinality.*

Degroupoidification works like this:
To *linearize* a (finite) groupoid, just take the free vector space on its space of isomorphism classes of objects, $\mathbb{C}^{\underline{A}}$.

Then there is a pair of linear maps associated to map $f : A \to B$:

- $f^* : \mathbb{C}^B \to \mathbb{C}^A$, with $f^*(g) = g \circ f$
- $f_* : \mathbb{C}^A \to \mathbb{C}^B$, with $f_*(g)(b) = \sum_{f(a)=b} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(a)} g(a)$

The first is just composition with $f$. The second is the map sending the vector $\delta_a$ to $\delta_{f(a)}$. These are adjoint with respect to an inner product such that $\langle [g_i], [g_j] \rangle = \frac{1}{\# \mathrm{Aut}(g_i)} \cdot \delta_{i,j}$.

This gives $D = t_* \circ s^*$ as a modified "sum over histories": when the groupoids are sets, this just counts the number of histories from $g_i$ to $g_j$. The general case counts with groupoid cardinality.

## Definition

The functor

$$D : Span(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$$

is defined by with $D(G) = \mathbb{C}(\underline{G})$, and

$$D(X)(f)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(x)} [f(s(x))]$$

In the case the groupoids are sets, this just gives multiplication by a matrix counting the number of histories from $x$ to $y$. In general, the matrix $D(X)$ has:

$$D(X)_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

The groupoid cardinality is a special case of the *volume of a stack*, which we need to deal with physically interesting examples.

### Definition

A **left Haar system** for a (loc.cpt.) groupoid **G** is a family $\{\lambda^x\}_{x \in G_0}$, where $\lambda^x$ is a (positive, regular, Borel) measure on $G^x = s^{-1}(x)$.

Unlike for Haar measure on a Lie group, a (left) Haar system $\lambda^x$ is not uniquely defined. It is only unique up to a (quasi-invariant, i.e. equivariant) measure $\mu$ on $M$.

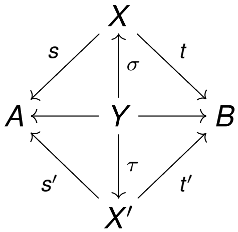
### Definition

If **G** is a groupoid, the space of objects is a measure space $(G_0, \mu)$, and $\lambda^x$ is a left Haar system, the *stack volume* of **G** is:

$$vol(\mathbf{X}) = \int_X \Big( \int_{s^{-1}(x)} d\lambda^x \Big)^{-1} d\mu$$

This is a stack invariant. (Based on Weinstein, where measures come from volume forms.)

Recall that the **2-morphisms** of $Span_2$(**Gpd**) are (iso. classes of) spans of *span maps*:




Composition is by weak pullback taken up to isomorphism.
Sometimes one just uses span maps: here, we want 2-morphisms as well as morphisms to have *adjoints*, and taking spans allows this.
We want a representation of $Span_2$(**Gpd**) that captures more than $D$, and preserves the adjointness property for both kinds of morphism.

## Definition

A finite dimensional **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear finitely semisimple abelian category (one with a “direct sum”, a.k.a. biproduct) generated by simple objects $x$, where $\hom(x,x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear (hence additive) functor. **2Vect** is the 2-category of KV 2-vector spaces, whose morphisms are 2-linear maps and whose 2-morphisms are natural transformations.

## Lemma

*If* $\mathbf{B}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{B}) = [\mathbf{B}, \mathbf{Vect}]$ *is a KV 2-vector space.*

The “basis elements” (generators) of $[\mathbf{B}, \mathbf{Vect}]$ are labeled by $([b], V)$, where $[b] \in \underline{\mathbf{B}}$ and $V$ an irreducible rep of $Aut(b)$.

## Definition

A 2-Hilbert space is an abelian $H^\star$-category.

Unpacking this definition, a 2-Hilbert space $H$ is an abelian category such that:

- each hom-set has the structure of a Hilbert space, and composition of morphisms is bilinear.
- $H$ is equipped with a star structure—a contravariant functor $* : H \to H$ which is the identity on objects and $*^2 = 1_H$.
- The star structure on $H$ induces an antinatural isomorphism

$$\hom(x, y) \cong (\hom(y, x))^*$$

In finite dimensions, this is much like **2Vect**, in that all 2-Hilbert spaces are equivalent to $\mathbf{Hilb}^n$, in which case 2-linear maps are equivalent to matrix multiplication with Hilbert space entries (using $\otimes$ and $\oplus$ in place of $+$ and $\times$).

Baez, Freidel et. al. conjecture the following for the infinite-dimensional case (incompletely understood):

## Conjecture

*Any 2-Hilbert spaces is of the following form:* **Rep($\mathcal{A}$)***, the category of representations of a von Neumann algebra $\mathcal{A}$ on Hilbert spaces. The star structure takes the adjoint of a map.*

## Example

*Rep*$(X)$ for a groupoid $X$, by taking $\mathcal{A}$ to be the completion of the groupoid $C^*$-algebra $C_c(X)$.

## Example

*Rep*$(L^\infty(X,\mu))$, for a measure space, gives the category of *measurable fields of Hilbert spaces* on $(X,\mu)$

In this context:

- For our physical interpretation $\mathcal{A}$ is the algebras of **symmetries** of a system. The algebra of **observables** will be its commutant - which depends on the choice of representation!
- Basis elements are irreducible representations of the vN algebra - physically, these can be interpreted as **superselection sectors**. Any representation is a direct sum/integral of these.
- Then 2-linear maps are functors, but can also be represented as **Hilbert bimodules** between algebras. The simple components of these bimodules are like matrix entries.

### Theorem

*If* $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \rightarrow \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \rightarrow \Lambda(\mathbf{X})$$

*namely composition with* $f$*, with* $f^* F = F \circ f$ *and*

$$f_* : \Lambda(\mathbf{X}) \rightarrow \Lambda(\mathbf{B})$$

*called "pushforward along* $f$*". Furthermore,* $f_*$ *is the two-sided adjoint to* $f^*$ *(i.e. both left-adjoint and right-adjoint).*

In fact, the adjoint map $f_*$ acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x)\cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is the left adjoint. But there is also a right adjoint:

$$f_!(F)(b) \cong \bigoplus_{[x]|f(x)\cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

In fact, this is a two-sided adjunction, by using the *Nakayama isomorphism*, a canonical isomorphism:

$$N_{(f,F,b)} : f_!(F)(b) \rightarrow f_*(F)(b)$$

given by the *exterior trace map* in each factor of the sum (which uses a modified group average).

Call the adjunctions in which $f_*$ is left or right adjoint to $f^*$ the *left and right adjunctions* respectively. We want to use the counit for the left adjunction, which is the evaluation map:

$$\eta_R(G)(x) : G(x) \to \bigoplus_{y|f(y)\cong x} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(y)], G(x))$$
$$v \mapsto \bigoplus_{y|f(y)\cong x} (g \mapsto g(v))$$

and the unit for the right adjunction, which just uses the action:

$$\epsilon_L(G)(x) : \bigoplus_{[y]|f(y)\cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} f^* G(x) \to G(x)$$
$$\bigoplus_{[y]|f(y)\cong x} g_y \otimes v \mapsto \sum_{[y]|f(y)\cong x} f(g_y)v$$

The Nakayama isomorphism does:

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

By composing units/counits with $N$, we get that $f^*$ and $f_*$ are ambidextrous adjoints.

Note: the group-average in $N$ is necessary to make this an isomorphism when working with modules over a general ring - not obvious working over $\mathbb{C}$!

## Definition

Define the 2-functor $\Lambda$ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\boldsymbol{a}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, we get that $\Lambda(X, s, t)$ is represented by the matrix with coefficients:

$$\Lambda(X, s, t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$

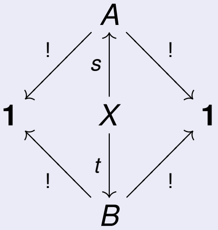
$$\simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

This is an intertwiner space, given by the *analog* of the inner product $\langle s^*\psi, t^*\phi\rangle$ in a Hilbert space.

In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

### Theorem

*Restricting to* $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*




*where* **1** *is the (terminal) groupoid with one object and one morphism, $\Lambda$ on 2-morphisms is just the degroupoidification functor $D$.*

The groupoid cardinality comes from the modified group average in $N$.

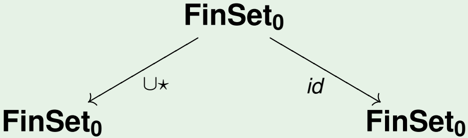
## Example

In the case where $\mathbf{A} = \mathbf{B} = \mathbf{FinSet_0}$ (equivalently, the symmetric groupoid $\coprod_{n \geq 0} \Sigma_n$ - note no longer finite), we find

$$D(\mathbf{FinSet_0}) = \mathbb{C}[[t]]$$

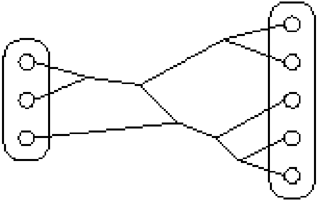
where $t^n$ marks the basis element for object $[n]$. This gets a canonical inner product and can be treated as the Hilbert space for the *quantum harmonic oscillator* ("Fock Space").

The operators $\mathbf{a} = \partial_t$ and $\mathbf{a}^\dagger = M_t$, generate the *Weyl algebra* of operators for the QHO. These are given under $D$ by the span $A$:



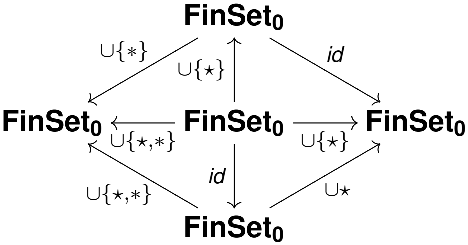

and its dual $A^\dagger$. Composites of these give a categorification of operators explicitly in terms of *Feynman diagrams*.

Such composites are described in terms of groupoids whose objects look like this:



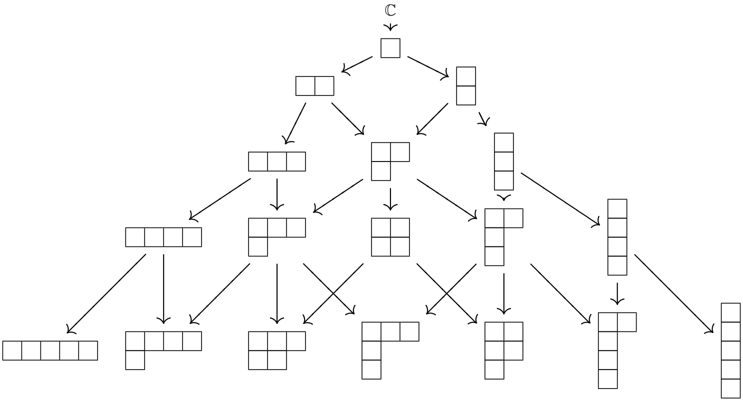
The source and target maps for the span pick the set of start and end points. The morphisms of the groupoid are graph symmetries. Degroupoidification $D$ calculates operators which (after small modification involving $U(1)$-labels) agree with the usual Feynman rules for calculating amplitudes.

An ongoing project (with Jamie Vicary) is to study the 2-categorical version of this picture. There are analogs of creation and annihilation operators in other *hom*-categories than *hom*$(1,1)$:




This is a 2-morphism $\alpha_A : A \to AAA^\dagger$ creates a “creation/annihilation pair” at the 1-morphism level.
Composites of these act as *rewrite rules* on the Feynman diagrams like those seen previously (now with “boundary” maps).

The image of this picture under $\Lambda$ involves representation theory of the symmetric groups as $\Lambda(\mathbf{FinSet_0}) \cong \prod_n Rep(\Sigma_n)$, and gives rise to "paraparticle statistics":



## Example

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Vect}$$

where $\mathbf{nCob_2}$ has

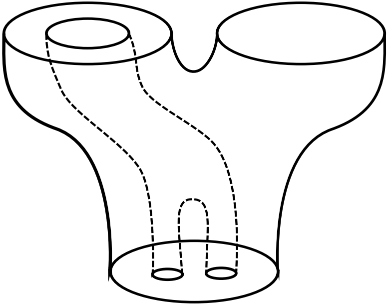
- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with $\partial M$ a union of source and target objects)
- **2-Morphisms**: $n$-dimensional cobordisms with corners

One construction uses *gauge theory*, for gauge group $G$ (here a finite group). Given $M$, the groupoid $\mathcal{A}_0(M,G) = hom(\pi_1(M), G)/\!\!/G$ has:

- **Objects**: Flat connections on $M$
- **Morphisms** Gauge transformations

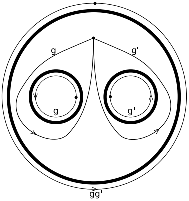
Then $\mathcal{A}_0(-,G) : \mathbf{nCob_2} \to Span_2(\mathbf{Gpd})$, and there is an ETQFT $Z_G = \Lambda \circ \mathcal{A}_0(-,G)$.

This relies on the fact that cobordisms in $\mathbf{nCob_2}$ can be transformed into products of cospans:

| $\mathbf{nCob_2}$ | $Span^2(Top)$ |
| --- | --- |
|  | $\begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A\coprod D) & \xleftarrow{i'_A\otimes i_D} & S^1\coprod S^1 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle i_2} \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle i_1} \\ S^1\coprod S^1 & \xrightarrow[i_2]{} & Y & \xleftarrow[i_1]{} & S^1 \end{array}$ |

Then $\mathcal{A}_0(-,G)$ maps these into $Span^2(\mathbf{Gpd})$.

Suppose $S : S^1 + S^1 \to S^1$ is the "pair of pants", showing two "particles" fusing into one.




Then we have the diagram:

$$\begin{array}{ccc} & (G \times G)/\!\!/G & \\ \swarrow^{\Delta} & & \searrow^{m} \\ (G/\!\!/G)^2 & & G/\!\!/G \end{array} \quad (4)$$

Where the map $\Delta$ leaves connections fixed, and acts as the diagonal on gauge transformations; and $m$ is the multiplication map.

- View $S^1$ as the boundary around a system (e.g. particle).
- Irreducible objects of $Z_G(S^1) \simeq [G /\!/ G, \mathbf{Vect}]$ are labelled by $([g], W)$, for $[g]$ a conjugacy class in $G$ and $W$ an irrep of its stabilizer subgroup
- For $G = SU(2)$, this is an angle $m \in [0, 2\pi]$, a particle; and an irrep of $U(1)$ (or $SU(2)$ for $m = 0$) is labelled by an integer $j$
- This theory then looks like 3D quantum gravity coupled to particles with mass and spin. with *mass* $m$ and *spin* $j$
- Under the topology change of the pair of pants, a pair of such reps is taken to one with nontrivial representations (superselection sectors) for all $[mm']$ for any representatives of $[m]$, $[m']$ (each possible total mass and spin for the combined system).

Dynamics (maps between Hilbert spaces) space arises from the 2-morphisms - componentwise in each 2-linear map.