Categorifying the Quantum Harmonic Oscillator

Jeffrey Morton
University of California, Riverside
morton@math.ucr.edu

June 15, 2006

Abstract

"Categorification" is the process of interpreting set-theoretic structures in mathematics as derived from category-theoretic structures. Examples include the interpretation of $\mathbb{N}[x]$ as Burnside rig of the category of combinatorial species. This has interesting applications to quantum mechanics, and in particular the quantum harmonic oscillator, via Joyal's "combinatorial species", and a new generalization called "stuff types" described by Baez and Dolan. Operators between stuff types be represented as rudimentary Feynman diagrams for the oscillator. In quantum mechanics, we want to represent states in an algebra over the complex numbers, and also want our Feynman diagrams to carry more structure than these "stuff operators" can do, and these turn out to be closely related. We will describe a categorification of the quantum harmonic oscillator in which the group of phases, $U(1)$, plays a special role. We describe a general notion of "M-stuff types" for any monoid M, and see that the case $M = U(1)$ provides an interpretation of time evolution in the combinatorial setting, and recover the usual Feynman rules for the oscillator.

Overview

- Understand the quantum **harmonic oscillator** by **categorifying** the Fock representation of the Weyl algebra (of operators on the space of states)
- Categorifying **Fock space** (a space of formal power series) gives **stuff types**. These have **generating functions** which are power series.
- There are also **stuff operators** which correspond to the operators on these power series
- Stuff types have a natural **inner product**, and we can use it to interpret **transition amplitudes** in terms of **Feynman diagrams**
- Generalizing to **M-stuff types** for any monoid M, and in particular the group of phases $U(1)$, describes of complex **amplitudes**

Quantum Harmonic Oscillator

Definition 1 ***Fock space*** *is the Hilbert space of states of the quantum harmonic oscillator. It can be described as a vector subspace of the space* $\mathbb{C}[\![z]\!]$ *of formal power series. The point* z^n *represents a "pure", state with energy* n*.*

The ***Weyl algebra*** *is the complex algebra generated by generators* $\boldsymbol{a}$ *(the* ***annihilation*** *operator) and* $\boldsymbol{a}^*$ *(the* ***creation*** *operator) satisfying the relations* $[\boldsymbol{a}, \boldsymbol{a}^*] = \boldsymbol{a}\boldsymbol{a}^* - \boldsymbol{a}^*\boldsymbol{a} = 1$*. It has a representation on Fock space where the generators are:*

$$\boldsymbol{a} f(z) = \frac{\mathrm{d}f(z)}{\mathrm{d}z}, \boldsymbol{a}^* f(z) = z f(z)$$

The space of states of the harmonic oscillator consists of the elements with finite norm with respect to an inner product on $\mathbb{C}[\![z]\!]$*. This is determined by its operation on the basis* $\{z^n\}$*, where* $\langle z^n | z^m \rangle = \delta_{n,m}\, n!$

This is so that $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are adjoint operators:

$$\begin{aligned} \langle z^n, z^n \rangle &= \left\langle \boldsymbol{a}^* z^{n-1}, z^n \right\rangle \qquad (1) \\ &= \left\langle z^{n-1}, \boldsymbol{a} z^n \right\rangle \\ &= n \cdot \left\langle z^{n-1}, z^{n-1} \right\rangle \end{aligned}$$

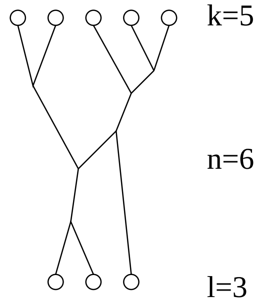
Definition 2

The inner product has the interpretation of a **transition amplitude** - a complex number related to the probability of going between two states. One is often interested in finding transition amplitudes like

$$\left\langle z^k | \phi^n z^l \right\rangle$$

where $\phi = (\boldsymbol{a} + \boldsymbol{a}^*)$.

Expressions like this involving polynomials in ϕ can be calculated using a sum over **Feynman diagrams**. These are diagrams of this general form:



We will categorify the algebraic structure for Fock space and the Weyl algebra in a way which makes the reason for this clear.

Stuff Types

Definition 3 *A* ***groupoid*** *is a category in which all morphisms are invertible. The* ***cardinality*** *of a groupoid* $\mathcal{G}$ *is*

$$|\mathcal{G}| = \sum_{[x]\in\overline{\mathcal{G}}} \frac{1}{|\operatorname{Aut}(x)|}$$

where $\bar{\mathcal{G}}$ *is the set of isomorphism classes of objects of* $\mathcal{G}$*. We call a groupoid* ***tame*** *if this sum converges.*

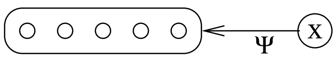
Groupoids provide a way of thinking about sets "divided by" the action of some group of symmetries (where morphisms come from group elements).

Definition 4 *A* ***stuff type*** *is a map (functor)* $\Psi : X \to \mathbf{FinSet_0}$*, where* X *is a groupoid, and* $\mathbf{FinSet_0}$ *is the groupoid of finite sets and their bijections. The* ***cardinality*** *or* ***generating function*** *of a stuff type is*

$$|\Psi| = \sum_{n\in\mathbb{N}} |\Psi_n| z^n$$

where Ψ_n *is the preimage of a finite set of size* n*, and* $|\Psi_n|$ *is its groupoid cardinality.*

We can think of a stuff type as a "vector" indexed by finite sets instead of numbers, or as a "bundle over $\mathbf{FinSet_0}$". We call the objects of the groupoid X **Ψ-stuffed finite sets** illustrated here:



The name comes from the fact that this generalizes **Ψ-structured finite sets** - equivalent to Joyal's **combinatorial species**.

U(1)-Stuff Types

To properly categorify *complex* power series, objects in a stuff type must occur with a certain phase in the group $U(1)$. We explain this for a general monoid M:

Definition 5 *If M is a monoid,* $\mathbf{MGpd}$ *is the category of M**-groupoids**. The objects are pairs (G, f), for $G \in$* $\mathbf{Gpd}$ *and $f : G \to M$, where the underlying set of M is regarded as a groupoid. Morphisms are commuting triangles:*

$$\begin{tikzcd} G_1 \arrow[r, "g"] \arrow[d, "f_1"'] & G_2 \arrow[dl, "f_2"] \\ M & \end{tikzcd} \tag{2}$$

*The **cardinality** of an M-groupoid $\mathcal{G}_M$ is an element of $\mathbb{R}^+ \otimes M$ given by the formal sum:*

$$|\mathcal{G}_M| = \sum_{[x] \in \underline{\mathcal{G}_M}} \frac{f(x)}{|\operatorname{Aut}(x)|} \tag{3}$$

Definition 6 *An M-stuff type is a functor $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$, where $\mathbf{X} \in \mathbf{MGpd}$. The **cardinality** of an M-stuff type $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$ is*

$$|\Psi| = \sum_{n \in \underline{\mathbf{FinSet_0}}} |\Psi_n| z^n \tag{4}$$

where $|\Psi_n|$ is now the M-groupoid cardinality of the preimage of n under Ψ.

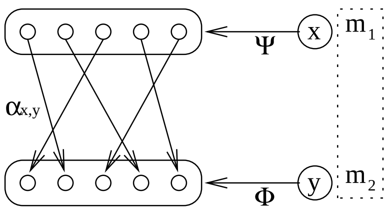
When $M = U(1)$, there is a homomorphism taking these cardinalities into $\mathbb{C}$ and $\mathbb{C}[\![z]\!]$, giving the **complex cardinality**.

Inner Product

There is an inner product on M-Stuff types (in particular, for stuff types, which simply have $M = 1$): given two stuff types, $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$ and $\Phi : \mathbf{Y} \to \mathbf{FinSet_0}$, the inner product is a groupoid $\langle \Phi, \Psi \rangle$, given as a **weak pullback** of Ψ and Φ. That is, a groupoid making the diagram

$$\begin{tikzcd} & \langle \Phi, \Psi \rangle \arrow[dl, "P_Y"'] \arrow[dr, "P_X"] & \\ \mathbf{Y} \arrow[dr, "\Phi"'] & & \mathbf{X} \arrow[ll, Leftarrow, "\alpha"] \arrow[dl, "\Psi"] \\ & \mathbf{FinSet_0} & \end{tikzcd} \tag{5}$$

commute up to a natural isomorphism α. Its objects are pairs of objects in the groupoids of $\mathbf{X}$ and $\mathbf{Y}$, labelled by the product of their labels, and with an isormorphism between the underlying sets. A typical object in $\langle \Psi, \Phi \rangle$ is:



Theorem 1 *The inner product agrees with cardinalities:*

$$|\langle \Phi, \Psi \rangle| = \langle |\Phi|, |\Psi| \rangle$$

Definition 7 *If* $\mathbf{X}$ *is a* $U(1)$*-groupoid and* $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$ *a* $U(1)$*-stuff type, the* ***complex cardinality*** *of* Ψ *is* h *applied to the usual* M*-stuff-type cardinality:*

$$|\Psi|_{\mathbb{C}} = h\left(\sum_{S\in\mathbb{N}} |\Psi_n| z^n\right) = \sum_{S\in\mathbb{N}} h|\Psi_n|z^n \tag{6}$$

where $|\Psi_n|$ *is the usual* M*-stuff cardinality,* h *is the above homomorphism, and addition is in* $\mathbb{C}$*.*

The **Fock space inner product** for $U(1)$-stuff types is:

$$\langle \Psi|\Phi\rangle = \left\langle \overline{\Psi}, \Phi \right\rangle \tag{7}$$

Theorem 2 *The Fock space inner product for* $U(1)$*-stuff types* Ψ *and* Φ *gives the usual conjugate-linear inner product on cardinalities* $\mathbb{C}[\![z]\!]$*, with:*

$$|\left\langle \Psi|\Phi\right\rangle|_{\mathbb{C}} = \left\langle |\Psi|_{\mathbb{C}} \middle| |\Phi|_{\mathbb{C}} \right\rangle$$

(This is because this groupoid is like a "sum" over all n of "products over n".)

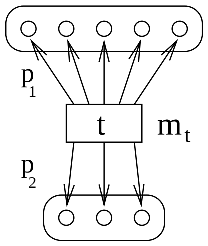
Remark 1 The complex conjugation in the Fock space inner product is related to the physical interpretation of the *bra* $\langle\psi|$ as "setting up" a state, and the *ket* $|\phi\rangle$ as "observing" a state. The inner product $\langle\psi|\phi\rangle$ is the *amplitude* to observe a state ϕ after setting up ψ. The squared modulus of the amplitude is the *probability*.

Stuff Operators

Definition 8 *An M**-stuff operator** is an M-groupoid T with two "projection" functors into* $\mathbf{FinSet_0}$ *(i.e. a span in* $\mathbf{MGpd}$*:*

$$\mathbf{FinSet_0} \xleftarrow{p_1} T \xrightarrow{p_2} \mathbf{FinSet_0}$$

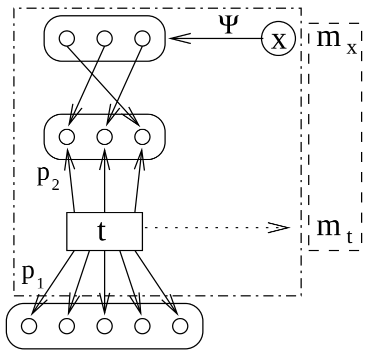
Objects of an M-stuff operator T look like this:



Definition 9 *There is also an* ***action of*** T ***on an*** M***-stuff type*** Ψ *giving a stuff type* $T\Psi$ *given by a weak pullback:*

$$\begin{tikzcd}
 & & T\Psi \arrow[dl, "P_T"'] \arrow[dr, "P_X"] \arrow[ddll, bend right, "p_1 \circ P_T"'] & \\
 & T \arrow[rr, Leftrightarrow, "\alpha"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & X \arrow[dl, "\Psi"] \\
\mathbf{FinSet_0} & & \mathbf{FinSet_0} &
\end{tikzcd} \tag{8}$$

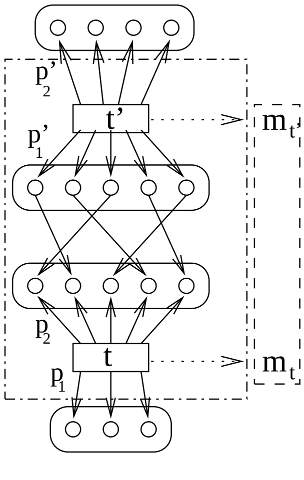
An object in $T\Psi$ *looks like:*



*A **composite** of T and T' given similarly:*

$$\begin{tikzcd}
 & & TT' \arrow[dl, "P_T"'] \arrow[dr, "P'_T"] \arrow[ddll, bend right, "p_1\circ P_T"'] \arrow[ddrr, bend left, "p'_2\circ P'_T"] & & \\
 & T \arrow[dl, "p_1"] \arrow[dr, "p_2"] \arrow[rr, Leftrightarrow, "\alpha"'] & & T' \arrow[dl, "p'_1"'] \arrow[dr, "p'_2"] & \\
\mathbf{FinSet_0} & & \mathbf{FinSet_0} & & \mathbf{FinSet_0}
\end{tikzcd} \tag{9}$$

Here is an object of the M-stuff operator $T'T$ (t is an object of T, labelled by m_t, and t' is an object of T', labelled by m_t). Its projections are p_1 and p'_2:



Composition of M-Stuff operators is similar to matrix multiplication relative to a basis indexed by $\mathbf{FinSet_0}$. Both M-stuff types and M-stuff operators can be added, and multiplied by arbitrary M-groupoids (i.e. they have a **linear structure**). These operations are defined using the **disjoint union** and **cartesian product** of groupoids, with the obvious M-labellings.

So M-stuff operators form a categorified algebra.

Feynman Diagrams (Stuff Types)

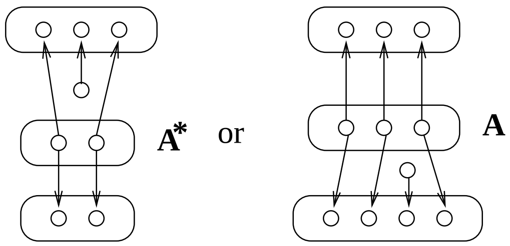
The categorified equivalents of the annihilation and creation operators $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are stuff operators called A and A^*. These are A:

$$\mathbf{FinSet_0} \xleftarrow{\mathbf{1}} \mathbf{FinSet_0} \xrightarrow{+\{\star\}} \mathbf{FinSet_0}$$

and its adjoint, A^*:

$$\mathbf{FinSet_0} \xleftarrow{+\{\star\}} \mathbf{FinSet_0} \xrightarrow{\mathbf{1}} \mathbf{FinSet_0}$$

So the objects look like:



Equipping a set S with $A\Phi$-stuff is just equipping $S \cup \{\star\}$ with Φ-stuff. Equipping S with $A^*\Phi$-stuff is just removing an element from S and equipping the resulting set with Φ-stuff.

Theorem 3 *The stuff type operators A and A^* satisfy the relation $A \circ A^* = A^* \circ A + 1$, and the cardinalities satisfy:*

$$|A\Phi| = \frac{\mathrm{d}}{\mathrm{d}z}|\Phi| = \boldsymbol{a}|\Phi|$$

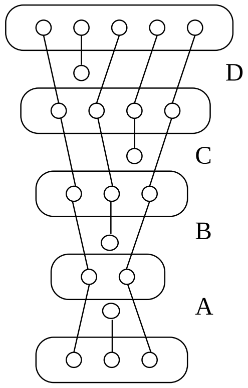
and

$$|A^*\Phi| = z|\Phi| = \boldsymbol{a}^*|\Phi|$$

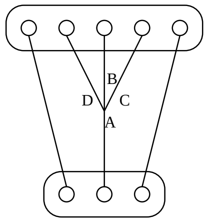
The first relation is true because there is one more way to remove an element from a set after first adding $\star$ to it than there is before.

Feynman Diagrams

We can use the stuff operators A and A^* to generate a "categorified algebra" of operators on stuff types. Powers of the field operator $\Phi = A + A^*$ are important. These have objects that are compositions of the A- and A^*-objects, as shown in this diagram (which suppresses all but the sets involved in the pullbacks):

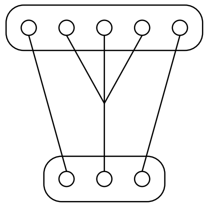


There is an isomorphism between these diagrams and ones of the following form, where some labelled strands are created or annihilated in a given order (shown is one in Φ^4):



There is an action of the permutation group on such a groupoid, changing the order in which the A and A^* are applied. A weak quotient by action gives $\Phi^k//S_k$, with the same objects as Φ^k, and whose morphisms are generated by adding morphisms for each permutation, taking an object to its image under that permutation.

The resulting groupoid is equivalent to one of diagrams like the above without labels.

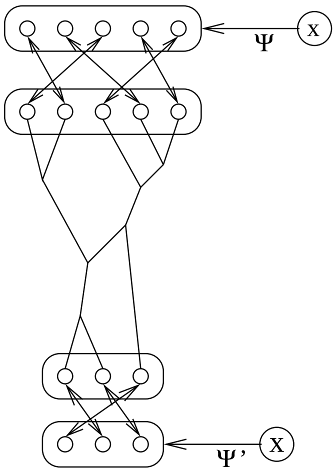


The groupoid cardinality gets along with the weak quotient of a groupoid by a group. For any groupoid C with a (functorial) action of G, $|C//G| = \frac{|C|}{|G|}$.

So when calculating things like $\langle \psi, p(\phi)\psi' \rangle$, we can treat this as the groupoid cardinality of the stuff type inner product

$$\langle \Psi, p(\Phi)\Psi' \rangle$$

with objects which look like this:

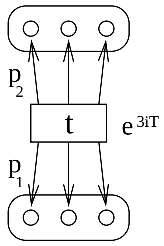


(The object in the middle is an abbreviated form of one in the stuff operator $(\Phi^4/\!\!/4!)^6$). The groupoid cardinality is a sum over all distinct objects like this, weighted by the size of various symmetry groups of these diagrams. This is exactly the Feynman rule, except that this does not incorporate phases.

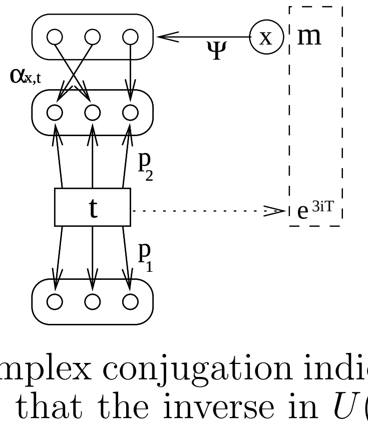
Time Evolution

The *free* quantum mechanical harmonic osciallator evolves in time by changing phase, at a frequency proportional to its energy. A state with k quanta of energy will change phase by e^{kiT} in the same time it takes for a state with one quantum of energy to change phase by e^{iT}.

Define, for any T, a $U(1)$-Stuff Operator, E_T, with two projections to $\mathbf{FinSet_0}$. Each object has two underlying sets of the same size, say k. The object with underlying sets of size k has weight



An object in the $U(1)$-stuff type $E_T\Psi$ amounts to an object from Ψ with its associated phase multiplied by e^{iNT} where N is the "energy" of the object: that is, the cardinality of its underlying set. This is shown below:



We see that complex conjugation indicates *time reversal*, from the fact that the inverse in $U(1)$ is an isomorphism.

Perturbation Theory

We've seen the time evolution for the free oscillator involves the operator $N = \boldsymbol{a}^*\boldsymbol{a}$. This is the *Hamiltonian*, H_0, for the free oscillator. The energy for the *perturbed* oscillator is modified by a "potential" term V, so $H = H_0 + V$, so the time evolution is given by e^{iHT}.

So the amplitude for finding a perturbed oscillator set up with energy k in a state with energy l after time T is then $|\left\langle Z^k | \mathrm{e}^{-iTH} Z^l \right\rangle|_{\mathbb{C}}$, which is:

$$\sum_{n=0}^{\infty} \int_{0 \le t_1 \le \cdots \le t_n \le T} \left| \left\langle Z^k | \mathrm{e}^{-i(T-t_n)N} V \mathrm{e}^{-i(t_n - t_{n-1})N} V \ldots V \mathrm{e}^{-it_1 N} Z^l \right\rangle \right|_{\mathbb{C}}$$

Typically, the potential V will be a polynomial in $\Phi = A + A^*$. In this case, except for the phase changes, this is analogous to our description of $\left\langle \psi | p(\frac{\phi^n}{n!}) \psi' \right\rangle$ as a sum over Feynman diagrams. In the case with time evolution, the sum is over all ways of breaking a time interval of length T into an alternating sequence of field operators and time evolution operators.

This shows how the Feynman rules express a groupoid cardinality (in the case with the infinite group $U(1)$ this involves the measure on $U(1)$, rather than the counting measure used in the discrete case).

Some References

This talk is based on *Categorified Algebra and Quantum Mechanics* by Jeffrey Morton:

`http://arxiv.org/abs/math.QA/0601458`

Other References

J. Baez, D. Wise. *"Categorification and Quantization"*. Lecture notes available at: `http://math.ucr.edu/home/baez/qg-fall2003/`
`http://math.ucr.edu/home/baez/qg-winter2004/`
`http://math.ucr.edu/home/baez/qg-spring2004/`

F. Bergeron, G. Labelle, P. Leroux. *Combinatorial Species and Tree-Like Structures: Encyclopedia of Mathematics and its Applications vol. 67*. Cambridge University Press, 1998.

Bob Coecke. *Kindergarten Quantum Mechanics - Lecture Notes*. Available as: `http://arxiv.org/abs/quant-ph/0510032`

M. Guta, H.Maassen. *Symmetric Hilbert Spaces Arising from Species of Structures*. 2000. `http://arxiv.org/abs/math-ph/0007005`

A. Joyal. Une théorie combinatoire des séries formelles. *Advances in Mathematics*, 42:1-82, 1981.

Richard D. Mattuck. *A Guide to Feynman Diagrams in the Many-Body Problem*. Dover Publications, Inc., 1967.