Groupoidification and 2-Linearization

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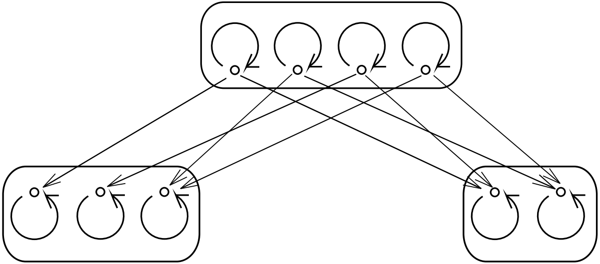


I will describe the category $Span_1(\mathbf{Gpd})$ and 2-category $Span_2(\mathbf{Gpd})$, and:

- "Degroupoidification", a functor $D : Span_1(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$
- "2-linearization", a 2-functor $\Lambda(:)Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$

Both of these generalize an obvious "linearization" functor

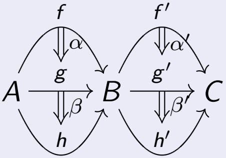
$$L : Span_1(\mathbf{Set}) \rightarrow \mathbf{Vect}$$

(Note: whenever convenient, assume all sets or groupoids here are finite).

Recall that a *category* is a structure with *objects*, and *morphisms* between objects, which have associative (partial) composition, and have unit morphisms for each object.

Definition

A *bicategory* **B** is a structure with objects, morphisms, and *2-morphisms*, which may be composed *vertically*, or *horizontally*:



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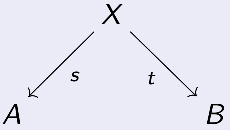
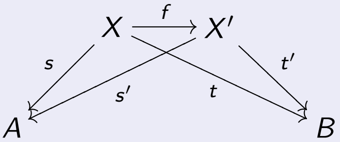
Associativity and unit laws are true up to *coherent 2-isomorphisms*. If these are identity 2-morphisms, **B** is a "*2-category*".

Definition

A **span** in a category **C** is a diagram of the form:

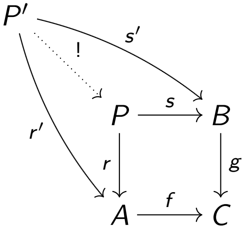


A *span map* f between two spans consists of a compatible map of the central objects:



A *cospan* is a span in $\mathbf{C}^{\text{op}}$ (i.e. **C** with arrows reversed).

Recall: in a category **C**, a *pullback* for a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ is an object P, with maps r and s into A and B which makes this square commute, and which is *terminal* for such objects:



(2)

It is defined up to (unique) isomorphism. If **C** has a pullback for every cospan, we say it *has pullbacks*.

Example

The category **Set** has pullbacks, given by the *fibred product*:

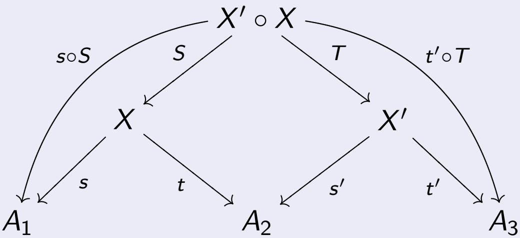
$$A \times_C B = \coprod_{c \in C} f^{-1}(c) \times g^{-1}(c) = \{(a,b) | f(a) = g(b)\}$$

If **C** is a category with pullbacks and products, we can define:

Definition

The category *Span*(**C**) has:

- **Objects**: Objects of **C**
- **Morphisms**: Isomorphism classes of spans in **C**
- Composition defined by pullback:



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- monoidal structure where $A \otimes B$ is the product in **C**, $A \times B$

To *linearize* a (finite) set, just take the free vector space on it, $\mathbb{C}^A$. Then there is a pair of linear maps associated to $f : A \to B$:

- $f^* : \mathbb{C}^B \to \mathbb{C}^A$, with $f^*(g) = g \circ f$
- $f_* : \mathbb{C}^A \to \mathbb{C}^B$, with $f_*(g)(b) = \sum_{f(a)=b} g(a)$

The first is just composition with f. The second is the map sending the vector δ_a to $\delta_{f(a)}$.

Using the standard inner product (such that A and B are orthonormal bases), these two maps are *adjoints*.

We can use these to define a functor:

Definition

For a set A, let $L(A) = \mathbb{C}^A$. Given a span $A \xleftarrow{s} X \xrightarrow{t} B$, define $L(X, s, t) = t_* \circ s^* : L(A) \to L(B)$.

So we have:

$$(L(X,s,t)(g))(b) = \sum_{t(x)=b} g(s(x))$$

This construction amounts to multiplication by a matrix with components:

$$L(X,s,t)_{a,b} = \#(s,t)^{-1}(a,b)$$

Theorem

This $L : Span(\mathbf{Set}) \rightarrow \mathbf{Vect}$ *is a monoidal functor.*

(In particular, it preserves composition:
$L(X \circ X', s \circ S, t' \circ T) = L(X,s,t) \circ L(X',s',t')$. "The fibred product instantiates matrix multiplication" by counting elements.)

Baez and Dolan described *groupoidification*, a way to extend the above to spans of groupoids.

Definition

A **groupoid** is a category in which all morphisms are invertible.

Groupoids describe "local symmetry" and simultaneously generalize sets and groups:

Example

- Any set S can be seen as a groupoid with only identity morphisms
- Any group G is a groupoid with one object
- Given a set S with a group-action $G \times S \longrightarrow S$ yields a transformation groupoid $S /\!\!/ G$ whose objects are elements of S; if $g(s) = s'$ then there is a morphism $g_s : s \longrightarrow s'$
- The category $\mathbf{FinSet_0}$ of finite sets and bijections is a groupoid
- An orbifold or smooth stack is represented by a (smooth) groupoid

Definition

There is a 2-category **Gpd** with:

- **Objects**: Groupoids
- **Morphisms**: Functors between groupoids
- **2-Morphisms**: Natural transformations between functors

Goal: Do the same for **Gpd** that we did for sets: convert groupoids to vector spaces and spans to linear maps.

Reason: The linear maps arising from *Span*(**Set**) are all represented by matrices with *positive integer* entries. Groupoids (and $U(1)$-*groupoids*) will allow us to capture more of linear algebra.

(Note: For the most part, all our groupoids will be finite, though many results generalize to infinite or smooth groupoids.)

Definition

The **cardinality** of a groupoid $\mathbf{G}$ is

$$|\mathbf{G}| = \sum_{[g] \in \underline{\mathbf{G}}} \frac{1}{\# \mathrm{Aut}(g)}$$

where $\underline{\mathbf{G}}$ is the set of isomorphism classes of objects of $\mathbf{G}$. We call a groupoid **tame** if this sum converges.

This has the nice property that it "gets along with quotients":

Theorem (Baez, Dolan)

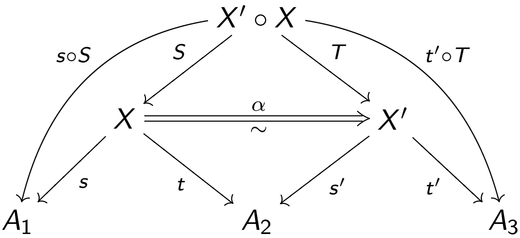
If S is a set with a G-action $G \times S \to S$, then

$$|S /\!\!/ G| = \frac{\# S}{\# G}$$

where $\#$ denotes ordinary set-cardinality.

The bicategory $Span_2(\mathbf{Gpd})$ has:

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- Composition defined by *weak* pullback:



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- **2-Morphisms** : isomorphism classes of *spans of* span maps
- monoidal structure from the product in **Gpd**

Note: This weak pullback of groupoids has objects (x, α, x'), where $\alpha : f(x) \rightarrow g(x')$, and its morphisms are commuting squares.

There is also a category $Span_1(\mathbf{Gpd})$, taking spans only up to isomorphism and neglecting the 2-morphisms, but still composing via weak pullback.

Using groupoid cardinality instead of set-cardinality, one can extend L to a functor:

$$D : Span_1(\mathbf{Gpd}) \to \mathbf{Vect}$$

For objects: $D(G) = H_0(G)$ (the zeroth homology $\mathbb{C}^{\underline{G}}$).
For morphisms, we modify the formula for sets:

$$D(X,s,t)(g)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(x)} [g(s(x))]$$

These come from two maps f^* and f_* as before, which are adjoint with respect to an inner product such that $\langle [g_i], [g_j] \rangle = \frac{1}{\# \mathrm{Aut}(g_i)} \cdot \delta_{i,j}$.

Definition

A **state** over a groupoid **G**, in $Span_1(\mathbf{Gpd})$, is (up to isomorphism) a span:

$$\mathbf{1} \stackrel{!}{\leftarrow} X \stackrel{\Psi}{\rightarrow} \mathbf{G}$$

The **cardinality** of a state is given by $D(X, !, \Psi)$ seen as a vector:

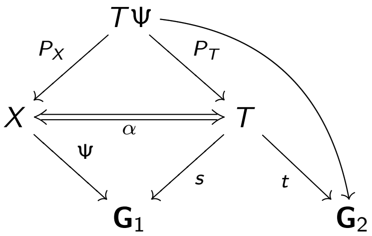
$$|\Psi| = \sum_{g \in \underline{\mathbf{G}}} |\Psi^{-1}(g)|[g]$$

where $|\Psi^{-1}(g)|$ is the groupoid cardinality of the *essential preimage* of g.

Example

In the case $\mathbf{G} = \mathbf{FinSet_0}$, a state is a (Baez-Dolan) "stuff type", which generalizes Joyal's "combinatorial species". Then Ψ is the "underlying set" functor, and objects of X are called "Ψ-stuffed finite sets" (or "Ψ-structured" when Ψ is faithful - i.e. when all morphisms in X are determined by those in $\mathbf{FinSet_0}$).

A span T in **Gpd** from $\mathbf{G}_1$ to $\mathbf{G}_2$ acts on a state Ψ over $\mathbf{G}_1$ by composition:



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Theorem (Baez, Dolan)

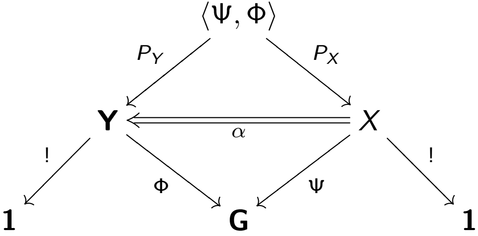
This action agrees with cardinalities:

$$|T\Psi| = |T||\Psi|$$

where $|T| = D(T, s, t)$ *is represented by the matrix with:*

$$|T|_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

There is an inner product on states: given two states, $\Psi : X \longrightarrow \mathbf{G}$ and $\Phi : Y \longrightarrow \mathbf{G}$, the inner product is a groupoid $\langle \Psi, \Phi \rangle$, given as a weak pullback. I.e. it is a composite of a state and costate as shown:



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Theorem (Baez, Dolan)

This "inner product" agrees with cardinalities:

$$|\langle \Phi, \Psi \rangle| = \langle |\Phi|, |\Psi| \rangle$$

To properly groupoidify *complex* vector spaces, we really need to use spans in $U(1)$-**Gpd**.

Definition

U(1)-Gpd is the category of $U(1)$-**groupoids**. The objects are pairs (G, f), for $G \in$ **Gpd** and f labels objects of G by elements of $U(1)$. The **cardinality** of a $U(1)$-groupoid $(\mathbf{G}, f)$ is the complex number:

$$|(\mathbf{G}, f)| = \sum_{[x] \in \underline{\mathbf{G}}} \frac{f(x)}{\# \mathrm{Aut}(x)} \tag{7}$$

Definition

A $U(1)$-state is given by $\Psi : \mathbf{X} \to \mathbf{G}$, where $\mathbf{X} \in$ **U(1)-Gpd**. The **cardinality** of an $U(1)$-state $\Psi : \mathbf{X} \to \mathbf{G}$ is

$$|\Psi| = \sum_{[g] \in \underline{\mathbf{G}}} |\Psi^{-1}(g)| [g] \tag{8}$$

(which comes from a $D_{U(1)}(\mathbf{X}, !, \Psi)$, analogous to the above).

Idea: Groupoid cardinality gives an equivalence relation on groupoids, which is coarser than isomorphism. (Unlike sets, where cardinalities are isomorphism classes). Degroupoidification gives invariants (up to equivalence) for groupoids and spans, but they lose some information. We'll describe a richer invariant: a (weak) 2-functor

$$\Lambda : Span_2(\mathbf{Gpd}) \to \mathbf{2Vect}$$

where $Span_2(\mathbf{Gpd})$ is a 2-category of *spans of groupoids* and **2Vect** is the 2-category of *Kapranov-Voevodsky* 2-vector spaces:

Definition

A **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear finitely semisimple additive category (one generated by simple objects x, where $\hom(x,x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear additive functor.

These, together with natural transformations between 2-linear maps, form a 2-category.

Theorem (Kapranov, Voevodsky)

Any 2-vector space is equivalent to $\mathbf{Vect^k}$ *(objects k-tuples of vector spaces, morphisms k-tuples of linear maps) for some k.*
Any 2-linear map $T : \mathbf{Vect}^k \rightarrow \mathbf{Vect}^l$ *is naturally isomorphic to a map of the form*

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^k V_{l,i} \otimes W_i \end{pmatrix}$$

Any natural transformation can be written as a matrix of linear maps between the components:

$$\begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,k} \\ \vdots & & \vdots \\ \alpha_{l,1} & \dots & \alpha_{l,k} \end{pmatrix} : \begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \longrightarrow \begin{pmatrix} V'_{1,1} & \dots & V'_{1,k} \\ \vdots & & \vdots \\ V'_{l,1} & \dots & V'_{l,k} \end{pmatrix}$$

Given a finite group G, the category $\mathbf{Rep}(G)$ has:

- **Objects**: Representations of G
- **Morphisms**: Intertwining operators between reps

Theorem

For any finite group G, $\mathbf{Rep}(G)$ is a 2-vector space

Any representation is a direct sum of irreducible reps - these form a *basis* for the 2-vector space.
By *Schur's Lemma*, if V_j is irreducible,

$$\hom(V_j, V_j) \cong \mathbb{C} \cdot 1$$

so these are indeed simple objects
We can make a similar construction for groupoids, since a group G is a one-object groupoid and:

$$\mathbf{Rep}(G) \cong [G, \mathbf{Vect}]$$

where we use the notation $[-, \mathbf{Vect}] = \hom(-, \mathbf{Vect})$.

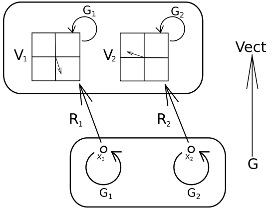
Lemma

If $\mathbf{G}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{G}) = [\mathbf{G}, \mathbf{Vect}]$ *is a KV 2-vector space.*

Note: If the automorphism groups of (isomorphism classes of) objects of $\mathbf{G}$ are $G_1, \dots, G_n$, then we have

$$[\mathbf{G}, \mathbf{Vect}] \cong \prod_j \mathbf{Rep}(\mathbf{G_j})$$

So the "basis elements" (simple objects) in $[\mathbf{G}, \mathbf{Vect}]$ are labeled by $([g], V)$, where $[g] \in \underline{\mathbf{G}}$ and V an irreducible rep of $Aut(g)$.



Given a span of groupoids:

$$\begin{array}{ccc} & X & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ A & & B \end{array}$$

we can apply the functor $[-, \mathbf{Vect}]$ to the whole diagram. This functor is contravariant, so we get a cospan:

$$\begin{array}{ccc} & [X, \mathbf{Vect}] & \\ {\scriptstyle s^*}\nearrow & & \nwarrow{\scriptstyle t^*} \\ \Lambda(A) & & \Lambda(B) \end{array}$$

Then $\Lambda(X, s, t)$ is given by a "pull-push" process with two stages.

Theorem (Morton)

If $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \rightarrow \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \rightarrow \Lambda(\mathbf{X})$$

namely composition with f*, with* $f^*F = F \circ f$ *and*

$$f_* : \Lambda(\mathbf{X}) \rightarrow \Lambda(\mathbf{B})$$

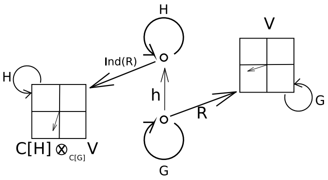
called "pushforward along f*". Furthermore,* f_* *is the two-sided (and 2-linear) adjoint to* f^**.*

In fact, the adjoint map

$$f_* : [\mathbf{X}, \mathbf{Vect}] \rightarrow [\mathbf{B}, \mathbf{Vect}]$$

to the composition acts on $F : \mathbf{X} \rightarrow \mathbf{Vect}$ to give the *induced representation*.

Given a group homomorphism $h : G \to H$, and a representation $R : G \to GL(V)$, there is an induced representation of H, namely $\mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$:



For our functor of groupoids, $f : \mathbf{X} \to \mathbf{B}$, we can push forward a representation in the same way. If more than one object is sent to the same $b \in \mathbf{B}$, we get a direct sum of all their contributions:

$$f_*(F)(b) \cong \bigoplus_{f(x) \cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a direct sum over the (essential) preimage of b in X.

Definition

For a span of groupoids $X : A \to B$ in $Span_2(\mathbf{Gpd})$ define the 2-linear map:

$$\Lambda(X,s,t) = t_* \circ s^* : \Lambda(A) \longrightarrow \Lambda(B)$$

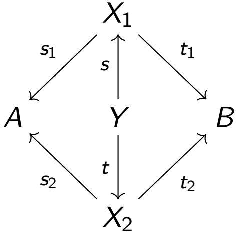
So then:

$$\Lambda(X,s,t)(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} (F \circ s)(x)$$

Picking basis elements $([a],V) \in \Lambda(A)$, and $([b],W) \in \Lambda(B)$, and using Frobenius reciprocity (i.e. the adjointness of our two 2-linear maps), we get that $\Lambda(X,s,t)$ is represented by the matrix:

$$\Lambda(X,s,t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$
$$\simeq \bigoplus_{[x]\in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

The most general 2-morphism for us is a span between spans, $Y : X_1 \longrightarrow X_2$ for $X_1, X_2 : A \longrightarrow B$:



we want a natural transformation

$$\Lambda(Y, s, t) : \Lambda(X_1, s_1, t_1) \longrightarrow \Lambda(X_2, s_2, t_2)$$

or

$$\Lambda(Y, s, t) : (t_1)_* \circ (s_1)^* \longrightarrow (t_2)_* \circ (s_2)^*$$

Up to scale, this can be constructed using the unit and counit for the adjunctions between t^* and t_*, and between s^* and s_*, and can be seen as another pull-push construction.

In coordinates:

$$\Lambda(Y,s,t)_{[([a],V),([b],W)]} : \bigoplus_{[x_1]} \hom_{Rep(\mathrm{Aut}(x_1))}[s_1^*(V), t_1^*(W)] \\ \rightarrow \bigoplus_{[x_2]} \hom_{Rep(\mathrm{Aut}(x_2))}[s_2^*(V), t_2^*(W)]$$

For each pair $([x_1],[x_2])$, this gives a map

$$\hom[s_1^*(V), t_1^*(W)] \rightarrow \hom[s_2^*(V), t_2^*(W)]$$

given by a sum over $[y] \in (s,t)^{-1}([x_1],[x_2])$.

To set the scale, we use the same weights as in $D : Span_1(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$.

Note: now we have spaces of intertwiners between representations of the groups $\mathrm{Aut}(x_i)$. With D, we only had a copy of $\mathbb{C}$ for each $[x_i]$.

Pulling a representation F back by f^* gives a representation of a new group, on the same space. An intertwiner is a linear map on this space which commutes with the representation.

So if $\phi : s_1^*(V) \rightarrow t_1^*(W)$ is an intertwiner in $Rep(\mathrm{Aut}(x_1))$ then the same underlying linear map is also an intertwiner $s^*(\phi) : (s_1 \circ s)^*(V) \rightarrow (t_1 \circ s)^*(W)$ in $Rep(\mathrm{Aut}(y))$. However, for the pushforward along t, the corresponding linear operator behaves as follows:

For $\phi \in \hom[s_1^*(V), t_1^*(W)]$ we get:

$$\Lambda(Y, s, t)_{[([a],V),([b],W)]}|_{([x_1],[x_2])}(\phi) = \frac{|(s,t)^{-1}(x_1, x_2)|}{|\,\mathrm{Aut}(x_2)|} \sum_{g \in \mathrm{Aut}(x_2)} g \phi g^{-1}$$

(This uses the *essential* preimage $(s,t)^{-1}(x_1, x_2)$ as before.)

The *group average* here projects a linear map into the space of intertwiners.

Given all this, the main fact is:

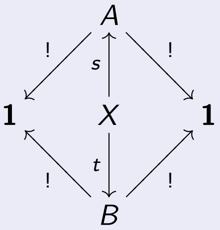
Theorem (Morton)

The process $\Lambda : Span_2(\mathbf{Gpd}) \to \mathbf{2Vect}$ *is a weak 2-functor.*

Furthermore:

Theorem (Morton)

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*



where $\mathbf{1}$ *is the (terminal) groupoid with one object and one morphism,* Λ *on 2-morphisms is just the degroupoidification functor* D*.*

"Physical" applications arise becase groupoids provide a good way of thinking about local symmetry.

In a span $A \leftarrow X \rightarrow B$, the groupoid X will represent a space of *histories*; s and t pick the starting and terminating *configuration* in spaces A and B.

We have suggested that the configuration spaces should be represented as groupoids, describing local symmetries. Often, these arise from global symmetries as quotients $S /\!\!/ G$. This is "doing physics in" the monoidal (2-)category $Span_2(\mathbf{Gpd})$.

Example

The degroupoidification functor D, in the case where $\mathbf{G} = \mathbf{FinSet_0}$, gives a space of formal power series in (say) t. This can be treated as the Hilbert space for the *quantum harmonic oscillator*.

Certain easy-to-describe spans describe the operators ∂_t and M_t, which generate an interesting algebra of operators - the *Weyl algebra*. Composing these produces *Feynman diagrams* explicitly.

An ongoing project (with Jamie Vicary) is to study the image of this picture under Λ. This involves representation theory of the symmetric groups.

Example

A Topological Quantum Field Theory (TQFT) is a monoidal functor

$$Z : \mathbf{nCob} \rightarrow \mathbf{Vect}$$

where **nCob** has:

- **Objects**: $(n-1)$-dimensional manifolds
- **Morphisms**: n-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)

They can be generated from *gauge theories*, given a group G. For a manifold M, there is a groupoid $\mathcal{A}_0(M, G)$ with:

- **Objects**: Flat connections on M
- **Morphisms** Gauge transformations

In fact, $\mathcal{A}_0(-, G) : \mathbf{nCob} \rightarrow Span_1(\mathbf{Gpd})$, and composing with D gives a TQFT $Z_G = D \circ \mathcal{A}_0(-, G)$.

Example

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Vect}$$

where $\mathbf{nCob_2}$ has

- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)
- **2-Morphisms**: n-dimensional cobordisms with corners

Then $\mathcal{A}_0(-,G) : \mathbf{nCob_2} \to Span_2(\mathbf{Gpd})$, and there is an ETQFT $Z_G = \Lambda \circ \mathcal{A}_0(-,G)$.
It happens to be the same as the "Dijkgraaf-Witten model" when $n=3$.