Groupoids, Spans, and Physics

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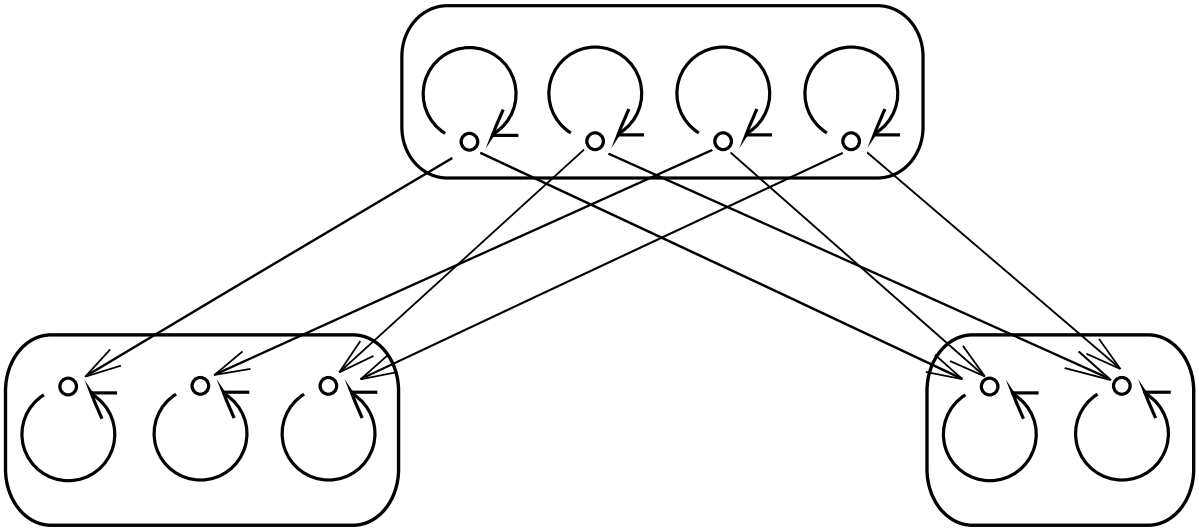


Abstract

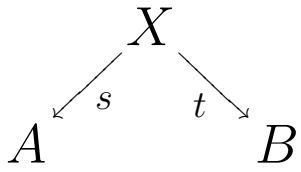
This talk will describe the program of "groupoidification": replacing the category of vector spaces and linear maps with the category Span(Gpd) of groupoids and spans between groupoids. I will describe this category, and the "degroupoidification" functor into Vect. In fact, Span(Gpd) is most naturally a bicategory: I will describe a related 2-functor, "2-linearization" from Span(Gpd) into 2Vect, the bicategory of Kapranov-Voevodsky 2-vector spaces. I will describe how both of these processes can be used to describe two "toy models" in quantum mechanics and quantum field theory.

Overview

A **groupoid** is a category in which all morphisms are invertible. Groupoids provide a good way of thinking about local symmetry. E.g. the transformation groupoid $S//G$ comes from a set S with an action of the group G: objects are elements of S, morphisms correspond to group elements.

Physically, groupoids can describe *configuration spaces* for physical systems. (Many physically realistic cases will also be, e.g. symplectic manifolds, whose points are the objects of the groupoid).

A **span** of groupoids is a diagram of the form:



Physically, X will represent a space of *histories*; s and t pick the starting and terminating *state* in configuration spaces A and B. (Note the system's configuration space can change.)

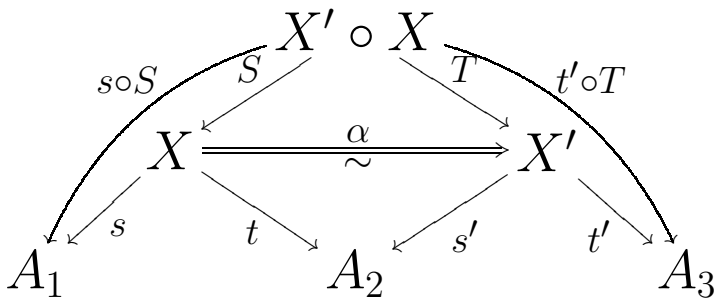
See [Mo1], [Mo2].

Span(Gpd)

This bicategory has:

• **Objects** - Groupoids

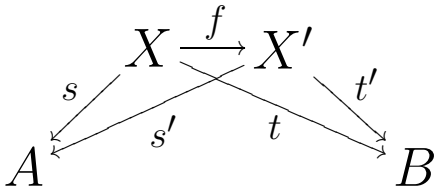
• **Morphisms** - Spans with composition defined by weak pullback:



(1)

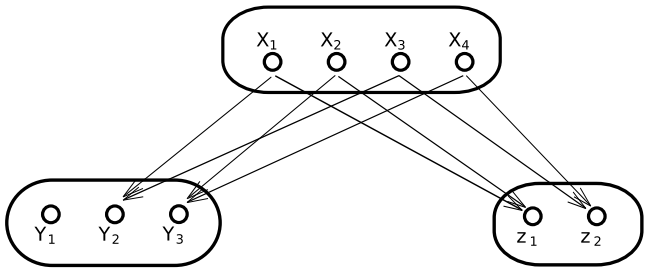
that is, a groupoid with projections making the above commute up to natural isomorphism α.

• **2-Morphisms**: Isomorphism classes of *spans of* span maps. A *span map* f between two spans consists of a compatible map of the central objects:



This bicategory has monoidal structure, and duals for morphisms and 2-morphisms.

There is an obvious way to get vector spaces and linear maps from spans of sets (linearize):



Take $L(A) = \mathbb{C}[A]$ and $L(B) = \mathbb{C}[B]$, free vector spaces on the sets A and B.

The linear map $L(X) : L(A) \to L(B)$, by "pulling and pushing" complex functions through the span:

$$(L(X)(f))(b) = \sum_{t(x)=b} f(s(x))$$

This construction amounts to multiplication by a matrix with components:

$$L(X)_{a,b} = |(s,t)^{-1}(a,b)|$$

Baez [HDA] described *degroupoidification*, a way to extend this to spans of groupoids. Groupoidification is a (nonrigorous) process to reverse this.

Degroupoidification

Definition 1 *The* ***cardinality*** *of a groupoid* $\mathcal{G}$ *is*

$$|\mathcal{G}| = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{|\operatorname{Aut}(x)|}$$

where $\bar{\mathcal{G}}$ *is the set of isomorphism classes of objects of* $\mathcal{G}$*. We call a groupoid* ***tame*** *if this sum converges.*

Using groupoid cardinality instead of set-cardinality, one can extend L to a 2-functor:

$$D : \underline{\operatorname{Span}(\mathbf{Gpd})} \to \mathbf{Vect}$$

with $D(G) = H_0(G)$ (the zeroth homology $\mathbb{C}(\underline{G})$),

$$D(X)(f)([b]) = |\operatorname{Aut}(b)| \sum_{[x] \in \underline{t^{-1}(b)}} \frac{[f(s(x))]}{|\operatorname{Aut}(x)|}$$

This amounts to multiplication by a matrix $D(X)$ with

$$D(X)_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

using groupoid cardinality.

Quantum Harmonic Oscillator

The *quantum harmonic oscillator* is a simple quantum mechanical system. Its "pure" states are indexed by nonnegative integer *energy levels*. A state with n "quanta" of energy has symmetry group S_n: quanta are indistinguishable (they are *bosons*).

There is a Hilbert space of states of the QHG, **Fock space**: $F \subset \mathbb{C}[\![z]\!]$, in which z^n represents a "pure", state with energy n.

The **Weyl algebra** acting on F is generated by

$$\boldsymbol{a} f(z) = \frac{\mathrm{d}f(z)}{\mathrm{d}z}, \boldsymbol{a}^* f(z) = z f(z)$$

These satisfy the *canonical commutation relation* $[\boldsymbol{a}, \boldsymbol{a}^*] = \boldsymbol{a}\boldsymbol{a}^* - \boldsymbol{a}^*\boldsymbol{a} = 1$.

The inner product on $\mathbb{C}[\![z]\!]$ is determined by the relation $\langle z^n | z^m \rangle = \delta_{n,m}\, n!$, so that $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are adjoint operators:

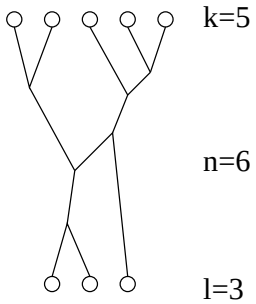
$$\begin{aligned} \langle z^n, z^n \rangle &= \left\langle \boldsymbol{a}^* z^{n-1}, z^n \right\rangle \\ &= \left\langle z^{n-1}, \boldsymbol{a} z^n \right\rangle \\ &= n \cdot \left\langle z^{n-1}, z^{n-1} \right\rangle \end{aligned} \tag{2}$$

The inner product has the interpretation of a **transition amplitude** - a complex number related to the probability of going between two states. One is often interested in finding transition amplitudes like

$$\left\langle z^k | \phi^n z^l \right\rangle$$

where $\phi = (\boldsymbol{a} + \boldsymbol{a}^*)$.

Expressions like this involving polynomials in ϕ can be calculated using a sum over **Feynman diagrams**. These are diagrams of this general form:



We will groupoidify Fock space and the Weyl algebra in a way which makes the reason for this clear.

Configuration Space

The groupoid describing the configuration space for the quantum harmonic oscillator has one object for each $n \in \mathbb{N}$, and object n has $\mathrm{Aut}(n) = S_n$, the symmetry group for n bosons.

This is equivalent (as a category) to the groupoid $\mathbf{FinSet_0}$, of finite sets and bijections. Since degroupoidification (and cardinality) are invariant under equivalence, we just call it $\mathbf{FinSet_0}$.

Everything we will see here for $\mathbf{FinSet_0}$ can be done for a general groupoid $\mathbf{G}$.

A *state* in quantum mechanics is a map into Hilbert space H from $\mathbb{C}$ (the monoidal unit).

Next we consider some constructions which are known to have physical meaning for the harmonic oscillator. These can all be done for any groupoid $\mathbf{G}$ in place of $\mathbf{FinSet_0}$.

Stuff Types

Definition 2 *A* ***stuff type*** *is a span*

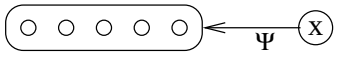
$$\mathbf{1} \leftarrow X \xrightarrow{\Psi} \mathbf{FinSet_0}$$

where X *is a groupoid, and* $\mathbf{FinSet_0}$ *is the groupoid of finite sets and their bijections. The* ***cardinality*** *or* ***generating function*** *of a stuff type is*

$$|\Psi| = \sum_{n \in \mathbb{N}} |\Psi_n| z^n$$

where Ψ_n *is the preimage of a finite set of size* n*, and* $|\Psi_n|$ *is its groupoid cardinality.*

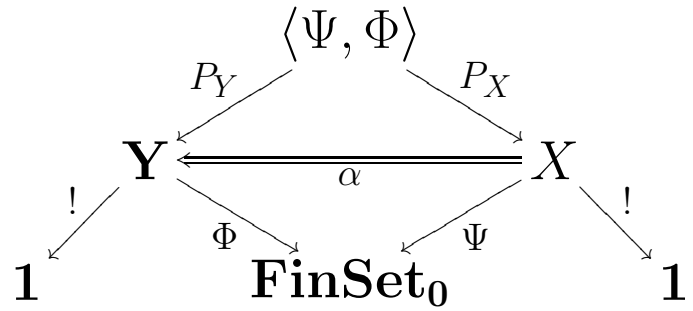
This gives the analog of a vector in Fock space. We call the objects of the groupoid X **Ψ-stuffed finite sets**:



This is a finite set, "equipped with Ψ-stuff". This terminology generalizes **structure types** - Joyal's **combinatorial species**, and equivalent to stuff types where Ψ is faithful.

Inner Product of Stuff Types

There is an inner product on stuff types (analogous to composing a vector and covector): given two stuff types, $\Psi : X \to \mathbf{FinSet_0}$ and $\Phi : Y \to \mathbf{FinSet_0}$, the inner product is a groupoid $\langle \Psi, \Phi \rangle$, given as a **weak pullback**. I.e. it is a composite of a vector and covector (shown):



(3)

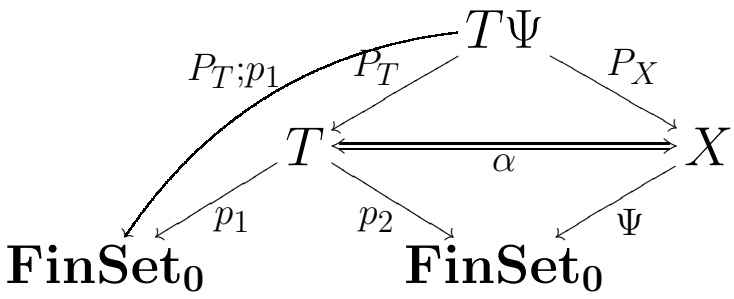
A typical object in $\langle \Psi, \Phi \rangle$ (up to iso.) is a triple (x, α, y), where $\alpha : \Psi(x) \to \Phi(y)$.

Theorem 1 *This inner product gets along with cardinalities:*

$$| \langle \Phi, \Psi \rangle | = \langle |\Phi|, |\Psi| \rangle$$

Stuff Operators

A **stuff operator** is a span in **Gpd** with $\mathbf{FinSet_0}$ as both source and target. This T acts on a stuff type Ψ by composition:



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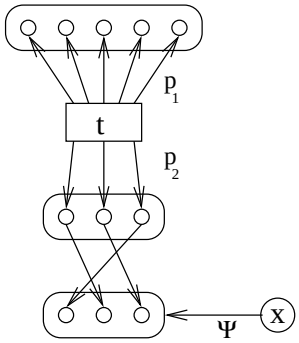
An object in $T\Psi$ looks like:



A **composite** of T and T' is also given by composition of spans.

The equivalents of the annihilation and creation operators $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are stuff operators called A and A^*. These are A:

$$\mathbf{FinSet_0} \xleftarrow{\mathbf{1}} \mathbf{FinSet_0} \xrightarrow{+\{\star\}} \mathbf{FinSet_0}$$

and its adjoint, A^*:

$$\mathbf{FinSet_0} \xleftarrow{+\{\star\}} \mathbf{FinSet_0} \xrightarrow{\mathbf{1}} \mathbf{FinSet_0}$$

Equipping a set S with $A\Phi$-stuff is just equipping $S \cup \{\star\}$ with Φ-stuff. Equipping S with $A^*\Phi$-stuff is just removing an element from S and equipping the resulting set with Φ-stuff.

Theorem 2 *The stuff type operators A and A^* satisfy the relation $A \circ A^* = A^* \circ A + 1$, and the cardinalities satisfy:*

$$|A\Phi| = \frac{\mathrm{d}}{\mathrm{d}z}|\Phi| = \boldsymbol{a}|\Phi|$$

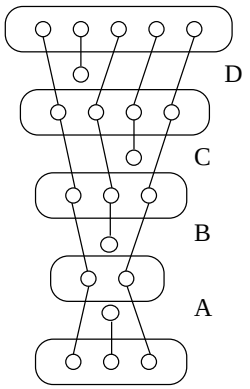
and

$$|A^*\Phi| = z|\Phi| = \boldsymbol{a}^*|\Phi|$$

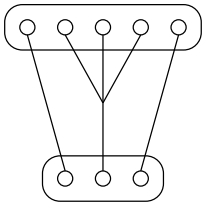
The first relation is true because there is one more way to remove an element from a set after first adding $\star$ to it than there is before.

Feynman Diagrams

Stuff operators A and A^* generate stuff operators by composition. Powers of the field operator $\Phi = A + A^*$ are important in quantum mechanics. These powers have objects with isomorphism classes labelled by diagrams like:



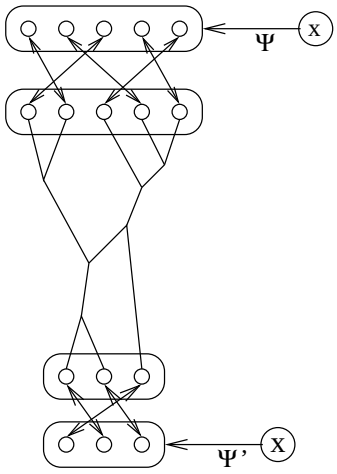
There is an action of S_k on the groupoid in the span Φ^k. A weak quotient gives $\Phi^k // S_k$, where the classes of objects have labels like:



To calculate $\langle \psi, p(\phi)\psi' \rangle$ (for a polynomial p), treat it as the groupoid cardinality of the stuff type inner product

$$\langle \Psi, p(\Phi)\Psi' \rangle$$

whose objects look like:



(The object in the middle is an abbreviated form of one in the stuff operator $(\Phi^3/\!\!/3!)^6$). It turns out these exactly describe the combinatorics of Feynman diagrams (a method for calculating amplitudes in quantum mechanics).

U(1)-Stuff Types

To properly groupoidify *complex* power series, objects in a stuff type must occur with a certain phase in the group $U(1)$.

Definition 3 **U(1)-Gpd** *is the category of* $U(1)$*-**groupoids**. The objects are pairs* (G, f)*, for* $G \in$ **Gpd** *and* f *labels (isomorphism classes of) objects of* G *by elements of* $U(1)$*. Morphisms are functors which are compatible with labellings.*

*The **cardinality** of a* $U(1)$*-groupoid* (G, f) *is the complex number:*

$$|(G, f)| = \sum_{[x] \in \underline{\mathcal{G}_M}} \frac{f(x)}{|\operatorname{Aut}(x)|} \tag{5}$$

Definition 4 *A* $U(1)$*-stuff type is a functor* $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$*, where* $\mathbf{X} \in$ **U(1)-Gpd**. *The **cardinality** of an* $U(1)$*-stuff type* $\Psi : \mathbf{X} \to \mathbf{FinSet_0}$ *is*

$$|\Psi| = \sum_{n \in \underline{\mathbf{FinSet_0}}} |\Psi_n| z^n \tag{6}$$

.

2-Linearization

Idea: Groupoid cardinality gives an equivalence relation on groupoids, which is coarser than isomorphism. (Unlike sets, where cardinalities are isomorphism classes). Degroupoidification gives invariants (up to equivalence) for groupoids and spans, but they lose some information.

We'll describe a richer invariant: a (weak) 2-functor

$$\Lambda : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{2Vect}$$

where $\mathrm{Span}(\mathbf{Gpd})$ is a 2-category of *spans of groupoids* and **2Vect** is the 2-category of *Kapranov-Voevodsky* 2-vector spaces, with:

- *objects*: 2-vector spaces
- *morphisms*: 2-linear maps
- *2-morphisms*: natural transformations

$\Lambda(\mathbf{G})$: 2-Vector Spaces

Definition 5 *A **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear finitely semisimple additive category (one generated by simple objects x, where $\hom(x,x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear additive functor.*

Theorem 3 *Any 2-vector spaces is equivalent to $\mathbf{Vect}^{\mathbf{k}}$ (objects k-tuples of vector spaces, morphisms k-tuples of linear maps) for some k.*

Any 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ is naturally isomorphic to a map of the form

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^k V_{l,i} \otimes W_i \end{pmatrix}$$

Any natural transformation can be written as a matrix of linear maps between the components.

Example

Given a finite group G, the category **Rep(G)** has:

• **Objects**: Representations of G

• **Morphisms**: Intertwining operators between reps

Theorem 4 *For any finite group* G, **Rep(G)** *is a 2-vector space*

Any representation is a direct sum of irreducible reps - these form a *basis* for the 2-vector space.

By *Schur's Lemma*, if V_j is irreducible,

$$\hom(V_j, V_j) \cong \mathbb{C} \cdot 1$$

so these are indeed simple objects

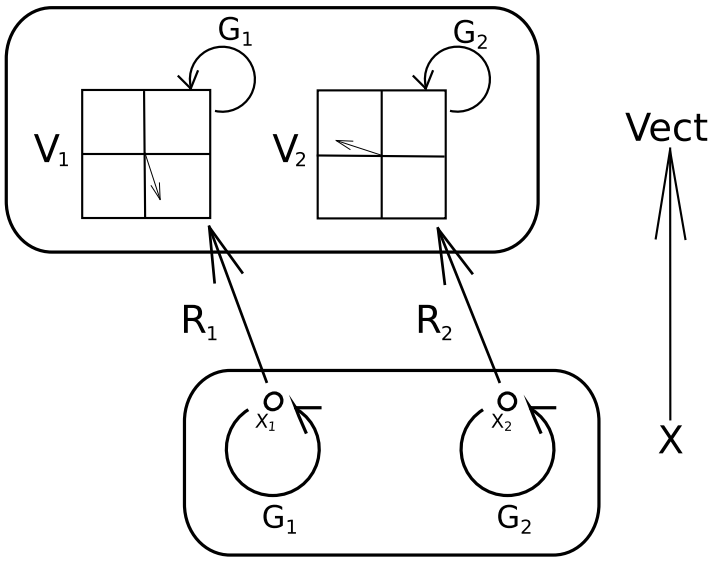
We can make a similar construction for groupoids as for groups. Taking a group G as a one-object groupoid:

$$\mathbf{Rep(G)} \cong [G, \mathbf{Vect}]$$

where we use the notation $[\mathbf{X}, \mathbf{Vect}] = \hom(\mathbf{X}, \mathbf{Vect})$.

Lemma 1 *If* $\mathbf{X}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{X}) = [\mathbf{X}, \mathbf{Vect}]$ *is a KV 2-vector space.*

Here is an illustration of a **Vect**-valued functor on $\mathbf{X}$:



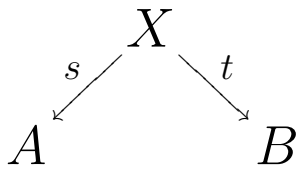
Note: If the automorphism groups of (isomorphism classes of) objects of $\mathbf{X}$ are $G_1, \ldots, G_n$, then we have

$$[X, \mathbf{Vect}] \cong \prod_j \mathbf{Rep}(\mathbf{G_j})$$

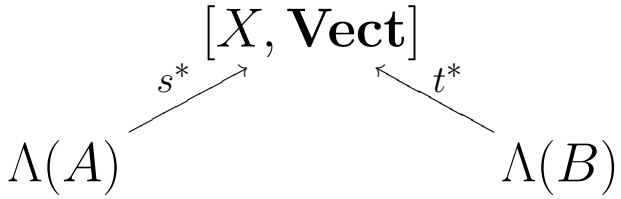
So the "basis elements" (simple objects) in $[X, \mathbf{Vect}]$ are labeled by $([x], V)$, where $[x] \in \underline{\mathbf{X}}$ and V an irreducible rep of $Aut(x)$.

$\Lambda(X)$: 2-Linear Maps for Spans

To push a 2-vector through a span of groupoids involves a (direct) sum over all "histories" joining specified endpoints:



we can apply the functor $[-, \mathbf{Vect}]$ to the whole diagram. This functor is contravariant, so we get a cospan:



Then $\Lambda(X)$ is given by a "pull-push" process with two stages.

Theorem 5 *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \to \mathbf{Y}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{Y}) \to \Lambda(\mathbf{X})$$

namely pullback along f*, with* $f^*F = F \circ f$ *and*

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{Y})$$

called "pushforward along f*". Furthermore,* f_* *is the two-sided (and 2-linear) adjoint to* f^**.*

Definition 6 *For a span of groupoids* $X : A \to B$ *in* $\mathrm{Span}(\mathbf{Gpd})$ *define the 2-linear map:*

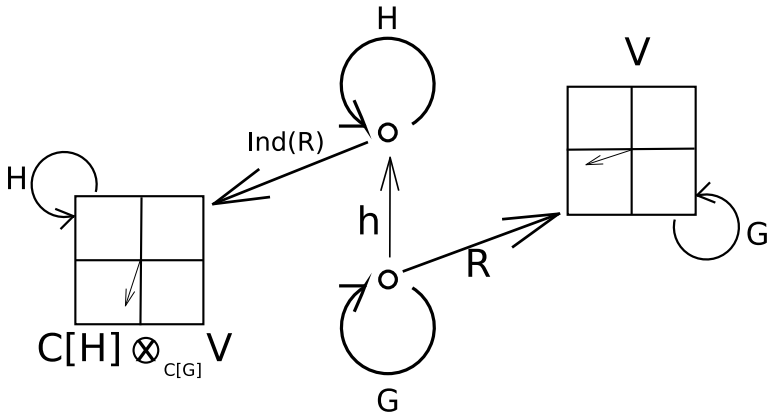
$$\Lambda(X) = t_* \circ s^* : \Lambda(A) \longrightarrow \Lambda(B)$$

In fact, the adjoint map

$$f_* : [X, \mathbf{Vect}] \to [Y, \mathbf{Vect}]$$

to the pullback acts on $f : \mathbf{X} \to \mathbf{Vect}$ to give the *induced representation.*

Given a group homomorphism $h : G \to H$, and a representation $R : G \to GL(V)$, there is an induced representation of H, namely $\mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$:



For a map of groupoids $t : X \to B$, we can push forward a **Vect**-presheaf in the same way. If more than one object is sent to the same $b \in B$, we get a direct sum of all their contributions. Assuming all groupoids are skeletal, this is:

$$t_*(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a direct sum over the (essential) preimage of $b \in B$.

So then:

$$\Lambda(X)(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} (F \circ s)(x)$$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, and using that

$$\mathbb{C}[G] \cong \bigoplus_i V_i \otimes V_i^*$$

where the sum is over irreps of G, Schur's lemma means that the 2-linear map $\Lambda(X)$ is represented by the matrix whose components are:

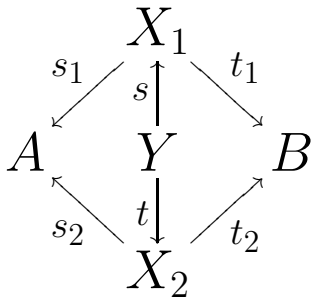
$$\begin{aligned} \Lambda(X)_{([a],V),([b],W)} &= \hom_{Rep(Aut(b))}[t_* \circ s^*(V), W] \\ &\simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}[s^*(V), t^*(W)] \end{aligned}$$

by adjointness (a.k.a. Frobenius reciprocity).

(Compare this to the degroupoidification formula. A cardinality has become a hom-space. This makes 2-morphisms possible!)

$\Lambda(Y)$: 2-Morphisms

The most general 2-morphism for us is a span between spans, $Y : X_1 \to X_2$ for $X_1, X_2 : A \to B$:



we want a natural transformation

$$\Lambda(Y) : \Lambda(X_1) \to \Lambda(X_2)$$

This is given as a matrix of linear operators between corresponding entries of $\Lambda(X_1)$ and $\Lambda(X_2)$:

$$\Lambda(Y)_{[([a],V),([b],W)]} : \Lambda(X_1)_{[([a],V),([b],W)]} \to \Lambda(X_2)_{[([a],V),([b],W)]}$$

This amounts to:

$$\Lambda(Y)_{[([a],V),([b],W)]} : \bigoplus_{[x_1]} \hom_{Rep(\mathrm{Aut}(x))}[s_1^*(V), t_2^*(W)] \\ \to \bigoplus_{[x_2]} \hom_{Rep(\mathrm{Aut}(x_2))}[s_2^*(V), t_2^*(W)]$$

such that for each block $([x_1], [x_2])$, the corresponding linear operator behaves as follows: for any intertwiner $f \in \hom[s_1^*(V), t_1^*(W)]$ we get:

$$\Lambda(Y)_{[([a],V),([b],W)]}|_{([x_1],[x_2])} = \frac{|\widehat{(x_1, x_2)}|}{|\mathrm{Aut}(x_2)|} \sum_{g \in \mathrm{Aut}(x_2)} gfg^{-1}$$

The sum here projects a linear map into the space of intertwining operators in the pushforward. This uses the groupoid cardinality of $\widehat{(x_1, x_2)}$, the essential preimage of (x_1, x_2) under (s, t).

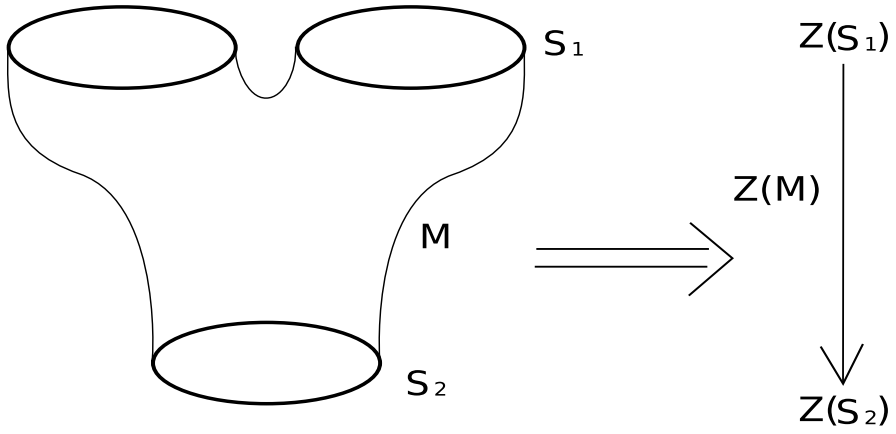
Given all this, the main fact is:

Theorem 6 *The process* $\Lambda : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{2Vect}$ *is a weak 2-functor.*

Application - Extended TQFT's

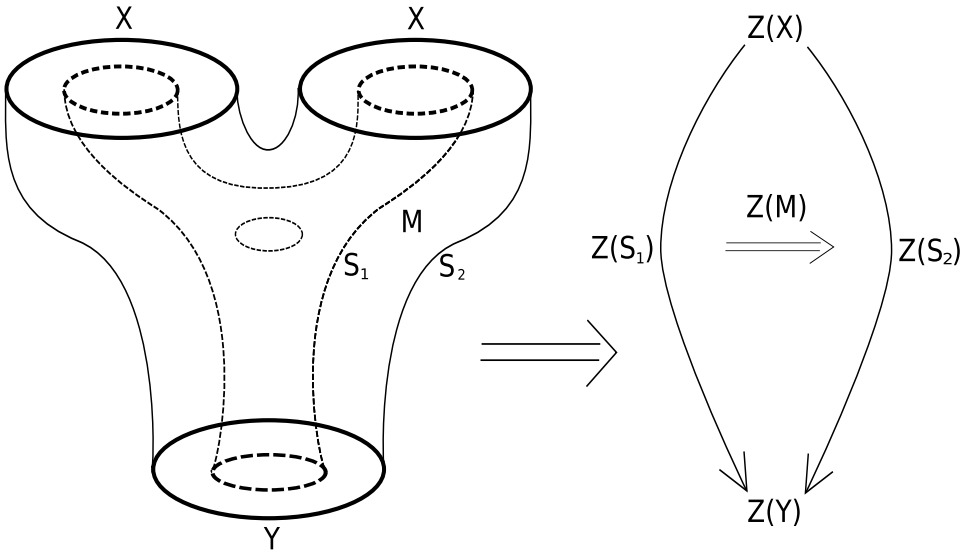
A Topological Quantum Field Theory is a monoidal functor:

$$Z : \mathbf{nCob} \to \mathbf{Vect}$$



An Extended TQFT is a (monoidal) 2-functor

$$Z_G : \mathbf{nCob}_2 \to \mathbf{2Vect}$$

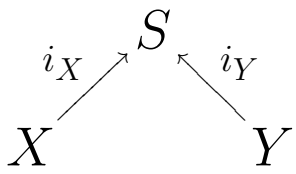


Cobordisms, Corners and Cospans

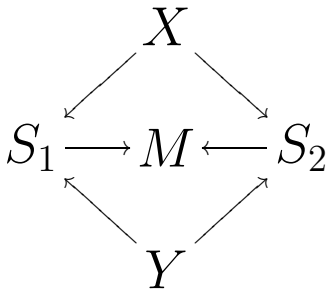
An *extended TQFT* has domain $\mathbf{nCob}_2$. This bicategory has:

- *Objects*: $(n-2)$-dimensional manifolds
- *Morphisms*: $(n-1)$-dimensional cobordisms
- *2-Morphisms*: (n)-dimensional cobordisms with corners

Cobordisms can be seen as cospans of manifolds (with corners) using inclusion maps:



Given a cobordism with corners between two cobordisms are double cospans, amounting to:



(Note there are complications due to the fact that $\mathbf{nCob}_2$ is a *cubical* weak 2-category.)

Z_G: Configuration Space

We use a gauge theory with finite gauge group G to define configuration spaces via an intermediate functor

$$\mathcal{A}_G : \mathbf{nCob}_2 \to \mathrm{Span}(\mathbf{Gpd})$$

to define "configuration" groupoids and spans of "histories", for manifolds and cobordisms, respectively.

The path groupoid $\Pi_1(B)$ of a manifold B consists of of points and homotopy classes of paths in B. A *flat G-connection* on a manifold B is an object in

$$\mathcal{A}_G(B) = \hom(\Pi_1(B), G)$$

Gauge transformations are natural transformations between these (giving a group element at each point).

Then the 2-vector space $Z_G(B)$ is:

$$Z_G(B) = \Lambda(\mathcal{A}_G(B))$$

Z_G: From Cobordisms to Histories

A "history" for the extended TQFT is a connection on the cobordism $S : X \to Y$ in $\mathbf{nCob}_2$. The maps s and t are just $(i_X)^\star$ and $(i_Y)^\star$: the restriction maps giving connections on the components of the boundary.

From a cobordism we get a span of the groupoids of *flat connections*:

$$\begin{array}{ccc} & \mathcal{A}_G(S) & \\ {\scriptstyle (i_X)^*}\swarrow & & \searrow{\scriptstyle (i_Y)^*} \\ \mathcal{A}_G(X) & & \mathcal{A}_G(Y) \end{array}$$

We use our weak 2-functor Λ to turn this into:

$$\Lambda(\mathcal{A}_G(X)) \xrightarrow{\Lambda(\mathcal{A}_G(S))} \Lambda(\mathcal{A}_G(Y))$$

and similarly for 2-morphisms.

Theorem 7 *For any finite group G, the 2-functor*

$$Z_G = \Lambda \circ \mathcal{A}_G$$

is an extended TQFT.

Example: $Z_G(S^1)$

For example, if $B = S^1$, $\Pi_1(S^1) \simeq \mathbb{Z}$ (in a "weak" sense). A G-connection g is determined by $g(1) \in G$. A natural transformation from g to g' is given by $h \in G$, such that $g' = hgh^{-1}$. So then:

$$\mathcal{A}_G(S^1) \simeq G/\!\!/G$$

which has:

- Objects: elements of G
- Morphisms: conjugacy relations $h : g \to g'$

In fact, it is eqivalent to take:

- Objects: conjugacy classes $[g]$ of G
- Morphisms: stabilizer subgroup of $[g]$

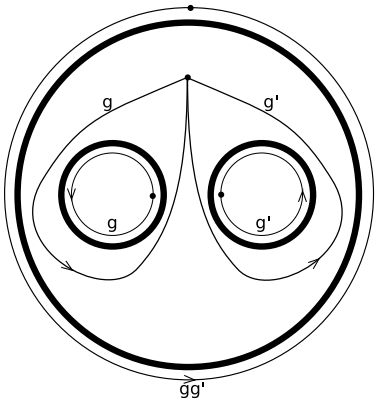
So we have the 2-vector space of G-*equivariant functors* into **Vect**, and thus

$$Z_G(S^1) = [\mathcal{A}_G(S^1), \mathbf{Vect}] \simeq [G/\!\!/G, \mathbf{Vect}]$$

The irreducible (basis) objects of $Z_G(S^1)$ are then labelled by a choice of conjugacy class $[g] \in \underline{G}$ and a representation $V \in Rep(Stab(g))$.

Example of $Z_G(S)$

Suppose $S : S^1 + S^1 \to S^1$ is the "pair of pants", showing two "particles" fusing into one.



Then we have the diagram:

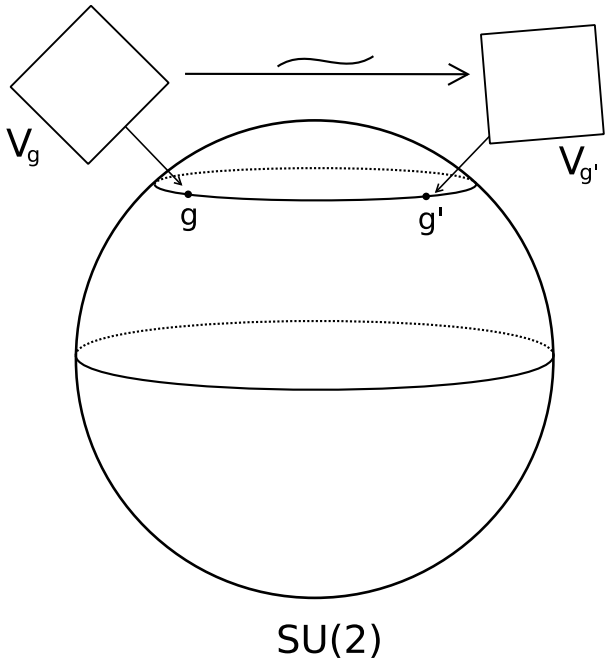
$$\begin{array}{ccc} & (G \times G)/\!\!/G & \\ {\scriptstyle \Delta}\swarrow & & \searrow{\scriptstyle m} \\ (G/\!\!/G)^2 & & G/\!\!/G \end{array} \tag{7}$$

Where the map Δ leaves connections fixed, and acts as the diagonal on gauge transformations; and m is the multiplication map.

A state over $([g], [g'])$ will be transported to one with nontrivial representations over all $[gg']$ for any representatives of $[g]$, $[g']$.

Goal: $G = SU(2)$

If $G = SU(2)$, the irreducible (basis) objects of $Z_G(S^1) \simeq [G/\!\!/G, \mathbf{Vect}]$ amount to a choice of conjugacy class in $SU(2)$ and representation of stabilizer, $U(1)$ ($SU(2)$ if $m = 0$):



View S^1 as the boundary around a point particle. A conjugacy class in SU(2) is an angle in $[0, 2\pi]$, which is the *mass* m of a particle; an irrep of $U(1)$ is labelled by an integer, the *spin* of a particle. This theory then looks like 3D quantum gravity with point particles with mass and spin.

Extension to Smooth Groupoids

What is different for smooth groupoids? First, the 2-vector space is generally infinite dimensional. One starting point is due to Crane and Yetter [meas]:

Definition 7 *If (X, μ) is a measurable space* **Meas(X)** *is the category with:*

- *Objects: measurable fields of Hilbert spaces on $(X, \mathcal{M})$: i.e. X-indexed families of Hilbert spaces $\mathcal{H}_x$ with a Hilbert space of measurable sections*
- *Morphisms: measurable fields of bounded linear maps between Hilbert spaces, $f_x : \mathcal{H}_x \to \mathcal{K}_x$ so that $||f||$ (the operator norm of f) is measurable.*

Then **Meas** is the 2-category of all categories **Meas(X)**, and we could look for a 2-functor

$$\Lambda : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{Meas}$$

(One should further do this for *2-Hilbert spaces* - using a measure μ to define a *direct integral*.)

What changes to Λ are needed in the infinite dimensional case?

The category of measurable functors $\mathcal{H} : \mathcal{G} \to \mathbf{Meas}$ amounts to a category of *equivariant* measurable fields of Hilbert spaces on the objects (so each $\mathcal{H}_x$ carries a representation of Aut(x).

• For a span $\mathcal{S} : \mathcal{G} \to \mathcal{G}'$, there will be a 2-linear map given by a *direct integral*:

$$(\Lambda(S)(\mathcal{H}))_y = \int^{\oplus} \mathcal{H}_x S_{([x],V),([y],W)} d\mu$$

Where the $S_{([x],V),([y],W)}$ are defined similarly to the above.

Generally, to extend both degroupoidification and 2-linearization to smooth groupoids:

• Finite product of $Rep(Aut(x_i)) \mapsto \mathbf{Meas}_{\mathcal{G}}(\mathcal{G}^{(0)})$

• Direct sum $\mapsto$ direct integral

• Counting measure $\mapsto$ measure on object space

• Groupoid cardinality $\mapsto$ volume of groupoid (cf. Weinstein [VS])

The algebraic part of the construction is the same...

Conclusion

We have suggested that the **configuration spaces** of physical systems should be represented as **groupoids**, describing local symmetries; and that the space of histories relating two systems through time as a **span** of groupoids. This is "doing physics in" the monoidal (2-)category Span(**Gpd**).

Degroupoidification is a functor turning this into physics in **Vect** (or **Hilb**), as usual in quantum mechanics. Span(**Gpd**) clarifies physics tools such as **Feynman diagrams**.

2-Linearization gives a more complete equivalence-invariant Λ for Span(**Gpd**). It provides a way to do physics in **2Hilb**. The **ETQFT** example shows how this can add a codimension to a topological invariant (also adds **matter** to a vacuum theory).

Both invariants rely on a ubitquitous **pull-push** process. Physically, this is Heisenberg's **matrix mechanics**.

Future projects: infinite-dimensional **2Hilb**; 2-linear QHO; ETQFT with Lie groups (3D QG); **U(1)-Gpd** version (action principle).

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