STACKS AND GROUPOIDS SEMINAR

SOME CHARACTERIZATIONS OF STACKS

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CONTENTS

1. Introduction

The goal of this lecture is to present some characterizations of stacks that have been used in previous lectures in the seminar. The characterizations show the nature of stacks: characterization 4.5 says that stacks are the categories where descent works, while characterization 4.6 means that we can regard a stack as a generalization of sheaves under the embedding of categories

$$\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C}) \to \mathbf{FibCat}(\mathbf{C}) \qquad \text{(see 3.12)},$$

so that proposition 4.7 and its corollaries 4.8 and 4.9 are the similar of proposition 2.3 and the resulting criterions for equivalence of sheaves 2.4 and 2.5, respectively.

For fixing notation purposes, we will recall the definitions of sheaves and stacks (as fibered categories).

1.1. **Notation.** Given a category $\mathbf{C}$, we will denote by $\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C})$ the category of presheaves (of sets) on $\mathbf{C}$. Given any object U in $\mathbf{C}$, we will denote by h_U the object in $\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C})$ represented by U.

2. Sheaves

2.1. Let U be an object in a category with pull-backs $\mathbf{C}$. A *sieve* S on U is a subfunctor of h_U.

A *(Grothendieck) topology* $\mathcal{T}$ on $\mathbf{C}$ consists of a collection of sieves $\mathrm{S}_\tau(U)$ for each object U in $\mathbf{C}$ such that

- GT1 h_U is in $\mathrm{S}_\tau(U)$;
- GT2 for any S in $\mathrm{S}_\tau(U)$ and any morphism $f : V \to U$, the subfunctor $f^{-1}(S) := \{\varphi : W \to V \mid f \circ \varphi \in S\}$ is in $\mathrm{S}_\tau(V)$;
- GT3 for any subfunctor R of h_U and any S in $\mathrm{S}_\tau(U)$ so that $f^{-1}(R)$ is in $\mathrm{S}_\tau(V)$ for all $f : V \to U$, R is in $\mathrm{S}_\tau(U)$.

The collections $\mathrm{S}_\tau(U)$ are called the *covering sieves* of the topology $\mathcal{T}$. The pair $(\mathbf{C}, \mathcal{T})$ will be called a *(Grothendieck) site*. For convinience, sometimes we will omit the topology $\mathcal{T}$ in the notation of a site.

We will consider only the sieves arising from coverings. Given any object U in $\mathbf{C}$, a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ induces the sieve on U defined as

$$\mathrm{h}_\mathcal{U}(V) := \{\phi : V \to U \mid \exists\, i \in I \text{ such that } \phi \text{ factors as } V \to U_i \to U\}$$

for V in $\mathbf{C}$. We will say that $\mathrm{h}_\mathcal{U}$ is the *sieve associated* to $\mathcal{U}$.

We say that a sieve $S \subset \mathrm{h}_U$ *belongs* to a topology arising from coverings if there is a covering $\mathcal{U}$ of U such that $\mathrm{h}_\mathcal{U} \subset S \subset \mathrm{h}_U$.

2.2. Let $\mathcal{F}$ be a presheaf on a site $\mathbf{C}$. Let U be an object in $\mathbf{C}$. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ consider the set defined by the equalizer diagram

$$\mathcal{F}(\mathcal{U}) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_i)\,.$$

The presheaf $\mathcal{F}$ is called a *sheaf* if the canonical map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is a bijection for all U in $\mathbf{C}$ and any covering $\mathcal{U}$ of U.

2.3. Proposition. Let $\mathcal{F}$ be a presheaf on $\mathbf{C}$. Let U be in $\mathbf{C}$. For any covering $\mathcal{U} = \{U_i \to U\}$ we have a commutative diagram

$$\begin{array}{ccc} \hom(\mathrm{h}_U, \mathcal{F}) & \xrightarrow[\cong]{\text{Yoneda}} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \hom(\mathrm{h}_{\mathcal{U}}, \mathcal{F}) & \xrightarrow{Y} & \mathcal{F}(\mathcal{U}) \end{array}$$

with the bottom arrow a bijection.

In this diagram, the vertical left arrow is induced by $\mathrm{h}_{\mathcal{U}} \subset \mathrm{h}_U$, and the vertical right arrow is the canonical map of 2.2.

The map Y is defined as follows: for φ in $\hom(\mathrm{h}_{\mathcal{U}}, \mathcal{F})$ set

$$Y(\varphi) = (\varphi(U_i \to U))_i \, .$$

Since $\varphi(U_i \to U) \in \mathrm{h}_{\mathcal{U}}(U_i)$, it follows that is well defined by universality of $\mathcal{F}(\mathcal{U})$. A direct computation shows that the diagram commutes.

The inverse of Y is defined as follows: for $(\alpha_i)_i \in \mathcal{F}(\mathcal{U})$, given $f : V \to U$ in $\mathrm{h}_{\mathcal{U}}(V)$, choose i such that f factors through $f_i : V \to U_i$ and set $Y^{-1}((\alpha_i)_i)$ as the map $\hom(V, U) \to \mathcal{F}(V)$, given by $f \mapsto f_i^*(\alpha_i)$.

2.4. Corollary. Under the hypothesis of 2.3, $\mathcal{F}$ is a sheaf if and only if $\hom(\mathrm{h}_U, \mathcal{F}) \to \hom(\mathrm{h}_{\mathcal{U}}, \mathcal{F})$ is a bijection.

2.5. Proposition. A presheaf $\mathcal{F}$ on $\mathbf{C}$ is a sheaf if and only if for all sieve S belonging to $\mathcal{T}$, the map

$$\hom(\mathrm{h}_U, \mathcal{F}) \longrightarrow \hom(S, \mathcal{F})$$

is a bijection.

Choose a covering $\mathcal{U}$ of U so that $\mathrm{h}_{\mathcal{U}} \subset S \subset \mathrm{h}_U$. We have the commutative diagram

$$\begin{array}{ccc} \hom(\mathrm{h}_U, \mathcal{F}) & \xrightarrow{f} & \hom(S, \mathcal{F}) \\ & \searrow^{h} & \downarrow g \\ & & \hom(\mathrm{h}_{\mathcal{U}}, \mathcal{F}) \, . \end{array}$$

Assume that $\mathcal{F}$ is a sheaf. By 2.4 we have that h is a bijection, so g is a surjection.

Now, let φ and ψ elements in $\hom(S, \mathcal{F})$ so that are equal under g. Any $V \to U$ in $S(V)$ induces $p_i : V \times_U U_i \to U$ in $\mathrm{h}_{\mathcal{U}}\,(V \times_U U_i)$, and from the assumption on φ and ψ we have

$$p_i^*\left(\varphi(V \to U)\right) = \varphi\left(V \times_U U_i\right) = \psi\left(V \times_U U_i\right) = p_i^*\left(\psi(V \to U)\right).$$

Conversely, since $\mathrm{h}_{\mathcal{U}}$ is a sieve, it follows from 2.4 that $\mathcal{F}$ is a sheaf.

3. Pseudo-Functors and Fibered Categories

3.1. A *pseudo-functor* Φ on a category $\mathbf{C}$ consists of the data:

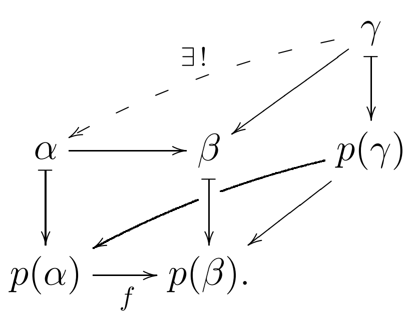
PF1 a category $\Phi(U)$ for each object U in $\mathbf{C}$;
PF2 a functor $\Phi(f) : \Phi(V) \to \Phi(U)$ for each map $f : U \to V$ in $\mathbf{C}$;
PF3 an isomorphism $\varepsilon_U : \Phi(\mathrm{Id}_U) \to \mathrm{Id}_{\Phi(U)}$ for each object U in $\mathbf{C}$;
PF4 an isomorphism $\Phi_{f,g} : \Phi(f)\Phi(g) \to \Phi(gf)$ for each composition $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathbf{C}$.

This data must satisfies certain axioms describing how to glue the compositions $U \to V = V$, $U = U \to V$, and $U \to V \to W \to W'$.

3.2. Let $p : \mathfrak{F} \to \mathbf{C}$ be a functor. For $\alpha \to \beta$ in $\mathfrak{F}$ and $U \to V$ in $\mathbf{C}$, that $p(\alpha \to \beta) = U \to V$ will be denoted by the diagram

$$\begin{array}{ccc} \alpha & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ U & \longrightarrow & V. \end{array}$$

A morphism $\alpha \to \beta$ in $\mathfrak{F}$ is *Cartesian* if for any $\gamma \to \beta$, and any map $g : p(\gamma) \to p(\alpha)$ in $\mathbf{C}$ so that $f \circ g = p(\gamma) \to p(\beta)$, there is a unique $\gamma \to \alpha$ mapped to g making commute the top face of the diagram



In such case we say that $f^*\beta := \alpha$ is the *pull-back* of β.

A *fibered category over* $\mathbf{C}$ is a functor $p : \mathfrak{F} \to \mathbf{C}$ where we can pull-back objects of $\mathfrak{F}$ along any map in $\mathbf{C}$.

A *morphism* of fibered categories over $\mathbf{C}$ is a functor $\phi : \mathfrak{F} \to \mathfrak{F}'$ such that we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{\ \phi\ } & \mathfrak{F}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathbf{C} & \end{array}$$

and ϕ maps Cartesian diagrams to Cartesian diagrams. The category of fibered categories over $\mathbf{C}$ will be denote by $\mathbf{FibCat}(\mathbf{C})$.

3.3. To any pseudo-functor Φ on $\mathbf{C}$ we can associate a fibered category $\mathfrak{F}_\Phi \to \mathbf{C}$:

- the objects of $\mathfrak{F}_\Phi$ are the pairs (α, U), where U is an object in $\mathbf{C}$ and α is in $\Phi(U)$.
- the morphisms are $(a, f) : (\alpha, U) \to (\beta, V)$, where $f : U \to V$ and $a : \alpha \to f^*\beta$.

The structure of category over $\mathbf{C}$ is given by

$$\begin{array}{c} \mathfrak{F}_\Phi \\ \downarrow \\ \mathbf{C} \end{array} \qquad\qquad \begin{array}{ccc} (\alpha, U) & \xrightarrow{(a,f)} & (\beta, V) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\ f\ } & V \end{array}$$

We see that is fibered by noticing that if $p(\beta) = V$, and $U \to V$ is any morphism in $\mathbf{C}$, then $(\mathrm{Id}_{f^*\beta}, f) : (f^*\beta, U) \to (\beta, V)$ is Cartesian.

3.4. Let $p : \mathfrak{F} \to \mathbf{C}$ be a fibered category. Let U be an object in $\mathbf{C}$. The *fiber* $\mathfrak{F}(U)$ is the subcategory of $\mathfrak{F}$ whose objects are mapped to U, and morphisms are the morphisms in $\mathfrak{F}$ mapped to Id_U.

By definition, a morphism of fibered categories $\phi : \mathfrak{F} \to \mathfrak{F}'$ induces a functor $\phi_U : \mathfrak{F}(U) \to \mathfrak{F}'(U)$ for any U.

3.5. Lemma. Let $\phi : \mathfrak{F} \to \mathfrak{F}'$ be a morphism of fibered categories. Then ϕ is an equivalence of categories if and only if ϕ_U is an equivalence of categories for all object U in $\mathbf{C}$.

3.6. A *cleavage* of a fibered category $\phi : \mathfrak{F} \to \mathfrak{F}'$ is a class C of Cartesian morphisms in $\mathfrak{F}$ such that for any $f : U \to V$ in $\mathbf{C}$ and for any β in $\mathfrak{F}(V)$, there is a unique morphism $\alpha \to \beta$ in C so that $p(\alpha \to \beta) = f$.

3.7. Proposition. (1) Every fibered category has a cleavage.

(2) The correspondence of 3.3 gives an equivalence between pseudo-functors and fibered categories with a cleavage. In this case, the corresponding pseudo-functor associated to a fibered category is given by the fiber (see 3.4).

3.8. Let

$$\begin{array}{ccc} \mathfrak{F} & \overset{F_1}{\underset{F_2}{\rightrightarrows}} & \mathfrak{G} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathbf{C} & \end{array}$$

be morphisms of fibered categories. We say that a natural transformation $f : F_1 \to F_2$ is *base preserving* if for all α in $\mathfrak{F}$ we have that the induced morphism $f_\alpha : F_1(\alpha) \to F_2(\alpha)$ is in $\mathfrak{G}(U)$, for $U := p(\alpha)$.

In such case, we say that f is an *isomorphism* if it is an isomorphism of functors.

Let $\hom_{\mathbf{C}}(\mathfrak{F}, \mathfrak{G})$ denote the category whose objects are morphisms of fibered categories, and morphisms are the base preserving natural transformations.

3.9. Let $\mathbf{FibCat}^{\mathbf{Set}}(\mathbf{C})$ be the full subcategory of $\mathbf{FibCat}(\mathbf{C})$ whose objects are the fibered categories $\mathfrak{F}$ for which $\mathfrak{F}(U)$ is a set for any object U in $\mathbf{C}$. In such case, the pull-back of an object in $\mathfrak{F}$ along any morphism in $\mathbf{C}$ is strictly unique. Thus, we have a presheaf

$$\Phi_{\mathfrak{F}} : \mathbf{C}^{op} \to \mathbf{Set}; \qquad U \mapsto \mathfrak{F}(U).$$

3.10. Proposition. The equivalence in 3.7 restricts to an equivalence between $\mathbf{FibCat}^{\mathbf{Set}}(\mathbf{C})$ and $\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C})$.

Explicitly, the inverse of the correspondence 3.9 is given by associating to each presheaf $\mathcal{F}$ the category fibered in sets whose objects are the pairs (U, α), with α in $\mathcal{F}(U)$, and the morphisms $f : (U, \alpha) \to (V, \beta)$ are given by morphisms $f : U \to V$ in $\mathbf{C}$ so that $\mathcal{F}(f)(\alpha) = \beta$.

From now on, we will denote by $\mathcal{F}$ the presheaf associated, under the equivalence of categories, to the fibered category on sets $\mathfrak{F}$.

3.11. Proposition. A fibered category $\mathfrak{F}$ over $\mathbf{C}$ is equivalent to $\mathbf{C}/X$, for some X in $\mathbf{C}$, if and only if $\mathfrak{F}$ is fibered in groupoids, and there is U in $\mathbf{C}$ and α in $\mathfrak{F}(U)$ such that for any ρ in $\mathfrak{F}$ there is a unique $\rho \to \alpha$ in $\mathfrak{F}$. In such case we say that $\mathfrak{F}$ is *representable.*

3.12. Corollary. A presheaf $\mathcal{F}$ is representable if and only if $\mathfrak{F}$ has terminal object. In particular, we have an embedding of categories

$$\mathbf{C} \longrightarrow \mathbf{FibCat}^{\mathbf{Gpd}}(\mathbf{C}); \qquad\qquad X \longmapsto \mathbf{C}/X \to \mathbf{C},$$

which extends the embedding $\mathbf{C} \to \mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C})$ given by Yoneda lemma.

We will denote by $\mathfrak{h}_X$ the fibered category $\mathbf{C}/X$, which corresponds to the presheaf h_X.

3.13. Remark. Notice that for any X, Y in $\mathbf{C}$ we have the bijection $\hom(X, Y) \to \hom(\mathbf{C}/X, \mathbf{C}/Y) = \hom(\mathfrak{h}_X, \mathfrak{h}_Y)$.

At the same time we have that if $\mathfrak{F}$ and $\mathfrak{G}$ are fibered categories equivalent to $\mathfrak{h}_X$ and $\mathfrak{h}_Y$, respectively, then we have an equivalence of categories $\hom(\mathfrak{h}_X, \mathfrak{h}_Y) \sim \hom(\mathfrak{F}, \mathfrak{G})$. The relevant part is that $\hom(\mathfrak{F}, \mathfrak{G})$ is not a set necessarily, but a equivalence relation at least.

3.14. 2-Yoneda. Let $\mathfrak{F}$ be a category fibered over $\mathbf{C}$. We have an equivalence of categories $\hom_{\mathbf{C}}(\mathfrak{h}_X, \mathfrak{F}) \to \mathfrak{F}(X)$.

The proof is the analogous of Yoneda's lemma.

4. Stacks

4.1. Let $\mathfrak{F}$ be in $\mathbf{FibCat}^{\mathbf{Gpd}}(\mathbf{C})$. Let $\mathcal{U} = \{U_i \to U\}$ be a covering in $\mathbf{C}$. An *object with descent data* on $\mathcal{U}$ is a collection of:

- objects α_i in $\mathfrak{F}(U_i)$,
- isomorphisms $\varphi_{ij} : pr_j^*\alpha_j \to pr_i^*\alpha_i$ on $\mathfrak{F}(U_i \times_U U_j)$,

satisfying the *cocycle condition*

$$pr_{ik}^*\varphi_{ik} = pr_{ij}^*\varphi_{ij} \circ pr_{jk}^*\varphi_{jk},$$

where $pr_i : U_i \times_U U_j \to U_i$ denotes the projection.

A *morphism* $\{f_i\} : (\alpha_i, \{\varphi_i\}) \to (\beta_i, \{\psi_i\})$ of objects with descent data is a map $f_i : \alpha_i \to \beta_i$ in $\mathfrak{F}(U_i)$ so that for any i, j, the diagram

$$\begin{array}{ccc} pr_j^*\alpha_j & \longrightarrow & pr_j^*\beta_j \\ \downarrow & & \downarrow \\ pr_i^*\alpha_i & \longrightarrow & pr_i^*\beta_i \end{array}$$

commutes. The category of objects with descent data on $\mathcal{U}$ will be denoted as $\mathfrak{F}(\mathcal{U})$.

4.2. For any covering $\mathcal{U} = \{\sigma_i : U_i \to U\}$ we have a correspondence

$$\mathfrak{F}(U) \longrightarrow \mathfrak{F}(\mathcal{U})$$

given by associating to each object α in $\mathfrak{F}(U)$ the object with descent data $\{\sigma_i^*\alpha, \psi_{i,j}\}$, where $\psi_{i,j} : pr_j^*\sigma_j^*\alpha \to pr_i^*\sigma_i^*\alpha$ are the isomorphisms granted by being $pr_j^*\sigma_j^*\alpha$ and $pr_i^*\sigma_i^*\alpha$ pull-backs of α to U_{ij}. It can be show that this correspondence is a functor independent of the choice of a cleavage.

4.3. A fibered category on groupoids $\mathfrak{F} \to \mathbf{C}$ is a *stack* if for any object U in $\mathbf{C}$ we have that the functor $\mathfrak{F}(U) \longrightarrow \mathfrak{F}(\mathcal{U})$ defined in 4.2 is an equivalence of categories.

4.4. Let $\mathcal{U} = \{U_i \to U\}$ be a covering in $\mathbf{C}$. We say that an object with descent data in $\mathfrak{F}(\mathcal{U})$ is *effective* if it is isomorphic to the image of an object of $\mathfrak{F}(U)$.

4.5. Characterization 1. A fibered category on groupoids $\mathfrak{F}$ over $\mathbf{C}$ is a stack if and only if for any covering $\mathcal{U} = \{U_i \to U\}$ in $\mathbf{C}$, the canonical functor $\mathfrak{F}(U) \to \mathfrak{F}(\mathcal{U})$ is fully faithful and all the objects with descent data in $\mathfrak{F}(\mathcal{U})$ are effective.

4.6. Characterization 2. Let $\mathfrak{F}$ be the fibered category of groupoids over $\mathbf{C}$ corresponding to the presheaf $\mathcal{F}$ under the embedding of categories $\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C}) \to \mathbf{FibCat}^{\mathbf{Gpd}}(\mathbf{C})$. Then $\mathfrak{F}$ is a stack if and only if $\mathcal{F}$ is a sheaf.

Fix a covering $\mathcal{U} = \{U_i \to U\}$ in $\mathbf{C}$. Notice that $\mathfrak{F}(U) = \mathcal{F}(U)$ as sets, and the objects of $\mathfrak{F}(\mathcal{U})$ are the collections of elements $(\alpha_i)_i$ so that $pr_i^*\alpha_i = pr_j^*\alpha_j$ on $\mathfrak{F}(U_i \times_U U_j)$. Thus, in this case the canonical functor 4.2 is given as $\alpha \mapsto (\alpha_{|U_i})$. The statement follows from the fact that to give an equivalence between discrete categories is equivalent to give a bijection between the underlying sets.

4.7. Proposition. Let $\mathfrak{F}$ be a fibered category on groupoids over $\mathbf{C}$. For any covering $\mathcal{U} = \{U_i \to U\}$ in $\mathbf{C}$ we have a commutative diagram

$$\begin{array}{ccc} \hom(\mathfrak{h}_U, \mathfrak{F}) & \xrightarrow[\cong]{\text{2-Yoneda}} & \mathfrak{F}(U) \\ \downarrow & & \downarrow \\ \hom(\mathfrak{h}_{\mathcal{U}}, \mathfrak{F}) & \longrightarrow & \mathfrak{F}(\mathcal{U}) \end{array}$$

with the bottom arrow is an equivalence of categories.

4.8. Characterization 3. Under the hypothesis of 4.7, $\mathfrak{F}$ is a stack if and only if $\hom(\mathfrak{h}_U, \mathfrak{F}) \to \hom(\mathfrak{h}_{\mathcal{U}}, \mathfrak{F})$ is an equivalence of categories.

4.9. Characterization 4. Let $(\mathbf{C}, \mathcal{T})$ be a site. A fibered category $\mathfrak{F}$ over $\mathbf{C}$ is a stack if and only if for all sieve S on an object U of $\mathbf{C}$ belonging to $\mathcal{T}$ we have an equivalence of categories

$$\hom_{\mathbf{C}}(\mathfrak{h}_U, \mathfrak{F}) \to \hom_{\mathbf{C}}(\mathfrak{S}, \mathfrak{F}),$$

where $\mathfrak{S}$ denotes the fibered category corresponding to the sieve S under the embedding of categories $\mathbf{Pshv}^{\mathbf{Set}}(\mathbf{C}) \to \mathbf{FibCat}^{\mathbf{Gpd}}(\mathbf{C})$.

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