In 1959 Grothendieck wrote a letter to Serre explaining that in trying to construct moduli spaces in algebraic geometry he keeps running into the problem that the underlying data has automorphisms

A concrete example

Why does the (fine) moduli space of elliptic curves not exist?

Lets work over $\mathbb{C}$.

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An *elliptic curve* is a smooth projective curve of genus 1 with a marked point.
As a complex manifold such a curve is a compact Riemann surface of the form $\mathbb{C}/\Lambda$ where Λ is a full sublattice of $(\mathbb{C}, +)$

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Lets denote the (non existant) moduli space by $\mathcal{M}$. There should be a universal family of elliptic curves $\mathfrak{U} \to \mathcal{M}$ with the following universal property:

Given any other family $\mathfrak{E} \to X$ of elliptic curves then there exists a unique morphism $f : X \to \mathcal{M}$ such that $f^*\mathfrak{U} \cong \mathfrak{E}$

This all falls apart when one inspects the family

$$\mathfrak{E} = \{(x, y, z) | y^2 = x(x-1)(x-z)\}.$$

where $\mathfrak{E} \to \mathbb{A}^1 \setminus \{0, 1\}$ with $(x, y, z) \mapsto z$.

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If we denote the fiber over t by $\mathfrak{E}_t$ then when checks that the curves $\mathfrak{E}_t$ and $\mathfrak{E}_{1-t}$ are isomorphic. This implies that the involution $z \mapsto 1 - z$ must lift to an involution of $\mathfrak{E}$. Some algebra shows that it can't.

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More conceptually the culprit is the fiber over $(1/2)$. It has an extra automorphism.

Conclusion : To have moduli spaces with all expected properties one needs a larger category of varieties. To see how one might construct such a category lets give a functorial characterization of algebraic varieties and schemes.

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One can characterise a variety (really a scheme) inside the functor category as being a sheaf that is locally representable.

Remark Lets rephrase the first example. Consider the functor

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This functor is a sheaf but it is not locally representable.

Towards a definiton of stack

The 2-category of stacks will be a certian subcategory of the 2-category of lax functors

$$\mathbf{Aff}^{\mathrm{op}} \to \mathbf{Gpds}.$$

The word lax is not been defined in this talk, we will illustrate its meaning with examples below before writing down a dictionary definition.

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If X is projective one can construct a relative version of BG called the moduli stack of G-bundles on X.

A *lax functor* $F : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Gpds}$ assigns a groupoid $F(\mathrm{Spec}(R))$ to each affine space $\mathrm{Spec}(R)$ and a pullback functor $f^* : F(\mathrm{Spec}(R)) \to F(\mathrm{Spec}(S))$ to every morphism $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Further there are natural isomorphisms

$$\alpha_{f,g} : f^* \circ g^* \xrightarrow{\sim} (gf)^*$$

This data is subject to the following constraints :

1. $\mathrm{id}^* = \mathrm{id}$
2. $\alpha_{f,\mathrm{id}} = \alpha_{\mathrm{id},g} = \mathrm{id}$
3. The following diagram commutes :

$$\begin{array}{ccc}
h^* \circ g^* \circ f^* & \xrightarrow{\sim} & h^* \circ (f \circ g)^* \\
\big\downarrow \wr & & \big\downarrow \wr \\
(g \circ h)^* \circ f^* & \xrightarrow{\sim} & (f \circ g \circ f)^*
\end{array}$$

Grothendieck topologies

Let **C** be a small category with finite fibered products. A Grothendieck topology on **C** consists of an assignment to each object X of **C** a collection $cov(X)$ of sets of arrows called the coverings of X such that

1. all isomorphisms are coverings, ie $Y \xrightarrow{\sim} X \in cov(X)$
2. $\{U_i \in cov(X)\}$ and $Z \to X$ implies $\{U_i \times_X Z\} \in cov(Z)$
3. if $\{U_i \to X\} \in cov(X)$ and $\{V_{ij} \to X_i\} \in cov(X_i)$ for each i, then $\{V_{ij} \to X_i \to X\} \in cov(X)$.

Examples

1. The usual topology on **Top**
2. The Zariski topology on **Aff**
3. The smooth or etale topology on **Aff**

Suppose that **C** has a topology. Further assume that **C** has coproducts.
A *sheaf* on **C** is a functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ such that for all such that $\{U_i \in \ cov(X)\}$ the following sequence is exact

$$F(X) \to F(U) \rightrightarrows F(U \times_X U)$$

A *stack* over **C** is a lax functor

$$F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Gpds}$$

such that

1. morphisms glue, more precisely for all $X \in \mathbf{C}$ and all $x.y \in F(X)$ the functor

$$\begin{array}{rcl} \mathrm{Isom}(x, y) & : & \mathbf{C}/X \to \mathbf{Sets} \\ (f : Y \to X) & \mapsto & \{\text{isomorphisms between } f^*x \text{ and } f^*y\} \end{array}$$

   is a sheaf.
2. objects glue, more precisely all descent data are effective.

Descent Data

Consider a covering family $\{U_i \to U\}$ an object $x \in F(U)$ produces via pullack objects $x_i \in F(U_i)$. Denote by $x_i|_{U_{ij}}$ the pullback of x_i to $F(U_i \times_U U_j)$. We have isomorphisms

$$\phi_{ij} : x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ji}}$$

subject to a cocylce condition on triple products.

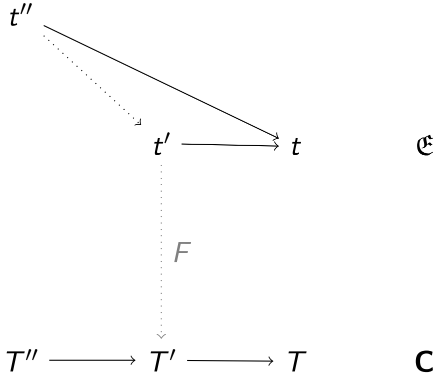
Such a family (x_i, ϕ_{ij}) is called a descent datum. The assertion that descent data are effective means that they all come from an x.

Stacks revisited

The category of stacks is in fact a 2-category but the 2-categorical stucture is not transparent from the above definition. In the literature it is customary to define a stack using categories fibered in groupoids. The defintion is equivalent. We recall it now.

A category fibered in groupoids over $\mathbf{C}$ is a category $\mathfrak{E}$ and a functor $F : \mathfrak{E} \to \mathbf{C}$ such that

1. for each morphism $f : y' \to y$ in $\mathbf{C}$ and $t \in \mathbf{C}$ with $F(t_y) = y$ there is a $t' \to t$ in $\mathfrak{E}$ projecting to f in $\mathbf{C}$.
2. for each diagram of the form



over $T'' \to T' \to T$ in $\mathbf{C}$ the triangle can be completed uniquely.

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A *stack* is a category fibered in groupoids such that arrows glue and all descent data are effective.

We have now a 2-category of stacks. This category has 2-carteasian products.

Algebraic Stacks

We work over the category **Aff** with the fppf topology. A morphism $F : X \to Y$ of stacks is said to be representable if for each scheme T and morphism $T \to Y$ the fibered product

$$T \times_Y X$$

is a scheme.

Let P be a property of morphisms of schemes that is invariant under base change. Then if $F : X \to Y$ is a representable morphism of stacks it makes sense to say that F has property P. If the $X \to X \times X$ is representable then all fibered products of schemes over X are schemes.

A stack X is said to be *algebraic* if the diagonal is representable quasicompact and seperated and there exists a smooth surjective morphism $S \to X$ where S is a scheme.
Remark Usually one replaces scheme be something more slightly general in the above defintion.

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This stack is presented by a point.(Use 2-Yoneda)

Given an algebraic stack with a presentation $P \to \mathfrak{X}$ we obtain a groupoid in schemes

$$P \times_{\mathfrak{X}} P \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} P$$

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2. One can prove highly non-trivial theorems about spaces using stacks.
3. Stacks can put a new perspective on existing constructions and theorems. For example, equivariant cohomology, ramified covers.