TOPOLOGICAL AND SMOOTH GROUPOIDS - WITH RELATIONS TO C^*-ALGEBRAS AND STACKS

JEFFREY C. MORTON

Abstract. In this talk, I will give some basic definitions and facts about topological and Lie groupoids. I will describe some examples involving group actions and configuration spaces of geometric structures. I will outline some work analogs of standard constructions in differential geometry, such as differential forms, in this context. I will also describe how these same examples can be described in terms of other kinds of structures: stacks (specifically, topological and differentiable stacks), and C^*-algebras associated to the groupoids. This talk will be a fairly introductory presentation of the subjects of this seminar.

0.1. **Groupoids Topological And Smooth.** The goal here is to sketch some interaction between three ideas which are used to deal with related examples: (1) topological and smooth groupoids, (2) C^*-algebras and W^*-algebras, and (4) topological and differentiable stacks.

Groupoids are structures that encode local symmetry. They are useful for treating equivariant problems as if they were not equivariant.

Definition 0.1.1. *A **groupoid** $\mathcal{G}$ (in* **Set***) is a category in which all morphisms are invertible. That is, as a category, consists of two sets G_0 (of objects) and G_1 (of morphisms) together with structure maps:*

$$s, t : G_1 \rightrightarrows G_0 \tag{1}$$

(the source and target maps) which are surjective, and

$$i : G_0 \to G_1 \tag{2}$$

(the identity inclusion) which is injective. There is a partially-defined associative multiplication:

$$\circ : G_1 \times_{G_0} G_1 \to G_1 \tag{3}$$

satisfying the usual properties (e.g. if $f : x \to y$, then $i(y) \circ f = f = f \circ i(x)$). All morphisms in G_1 are invertible, so we have:

$$(-)^{-1} : G_1 \to G_1 \tag{4}$$

satisfying the usual properties of an inverse.

A topological groupoid is a groupoid in **Top***: all the sets above are topological spaces and the structure maps are continuous functions. A smooth groupoid is a groupoid in* **Diff***: all sets are smooth (i.e. C^∞) manifolds and maps are smooth (also: s, t are epi, so they are surjective submersions)*

In the smooth case, it is usual to write $G = G_1$ and $M = G_0$. Smooth groupoids could be thought of as "bibundles", since there are two bundles $G \rightrightarrows M$. They could also be seen as (smooth) graphs, since any category is a directed graph, and any groupoid an undirected one. Alternatively, we could describe everything in terms of G_1, since G_0 embeds in it as a subspace/submanifold consisting of the units in G_1.

Example 0.1.2. There are many common examples of groupoids:

- A group G can be seen as a groupoid with $M = \star$.
- Given a group action $a : G \times X \to X$, the **transformation groupoid** $X /\!\!/ G$ is:

$$G \times X \rightrightarrows X \tag{5}$$

  with $s = \pi_X$ and $t = a$. Then $M = X$, and $G = \{(g, x) : x \mapsto g(x)\}$.
- Given a category $\mathcal{C}$, ignoring all but its isomorphisms gives a groupoid. For example, **FinSet** contains $\mathbf{FinSet_0}$.
- **Orbifolds**: like manifolds, but locally homeomorphic to $\mathbb{R}^n/G$, where G is a discrete group acting on $\mathbb{R}^n$. (sketch of teardrop orbifold example)
- Given a differentiable manifold M, the **fundamental groupoid** $\Pi_1(M)$ which has objects $x \in M$ and morphisms homotopy classes of paths in M.
- Given a differentiable manifold M and Lie group G, the groupoid $\mathcal{A}_G(M)$ of **principal G-bundles** and bundle maps; and the groupoid $\mathcal{A}_{G,flat}(M)$ of FLAT G-bundles and maps.

The last is an example of a geometric *moduli space* being considered as a groupoid, since if M is connected, the space G_0 is the usual moduli space. The morphisms (bundle maps) come from an action of a global symmetry group (the *gauge group*). Groupoids can be built by gluing (locally) quotient groupoids together to describe *local symmetry*.

Since groupoids are categories, it is usual to think of them up to *equivalence* (the weaker notion of isomorphism), to treat different but "indistinguishable" groupoids as the same:

Definition 0.1.3. *Two categories (say groupoids $\mathcal{G}$ and $\mathcal{G}'$) are* ***equivalent*** *if there are functors $f : \mathcal{G} \to \mathcal{G}'$ and $g : \mathcal{G}' \to \mathcal{G}$ with $g \circ f \simeq \mathrm{Id}_{\mathcal{G}}$ and $f \circ g \simeq \mathrm{Id}_{\mathcal{G}'}$.*

Proposition 0.1.4. *Any category (hence $\mathcal{G}$) is equivalent to its skeleton $\underline{\mathcal{G}}$, which has:*

- *one object for each isomorphism class of $\mathcal{G}$*
- *the only morphisms are $Aut([x]) \cong Aut(x)$ for each $[x] \in \mathcal{G}$*

Example 0.1.5. If $\mathcal{G} = X /\!\!/ G$ is the transformation groupoid of a group action, $\underline{\mathcal{G}}$ has space of objects X/G, the usual quotient space. The only morphisms are isomorphic copies of the groups $Aut(x)$, the stabilizer subgroups of G fixing x.

Topologically, this suggests a space whose points are decorated by the groups $Aut(x)$ - note, however, that the equivalence might not be implemented by continuous/smooth functions! That is, $\mathcal{G}$ might not be equivalent *as a topological/smooth groupoid* to $\underline{\mathcal{G}}$. For topological and smooth groupoids, it is usual to weaken this to a homotopical notion:

Definition 0.1.6. *Two groupoids $\mathcal{G}$ and $\mathcal{G}'$ are **Morita equivalent** if there is a pair of morphisms:*

(6)
$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{G} & & \mathcal{G}' \end{tikzcd}$$

where both f and g are "Morita morphisms" (specifically, these are local acyclic fibrations - in particular, it preserves on hom-sets $hom(x, y)$ up to isomorphism).

Many useful constructions with groupoids are invariant under Morita equivalence. It is also related to representation theory of the groupoids (we return to this in the next section).

Typical questions about Lie groupoids combine those usually asked about manifolds (differential geometry) and Lie groups (e.g. representation theory).

For example, there is the equivalent of the Lie algebra of a Lie group G.

Definition 0.1.7. *A **Lie algebroid** on manifold M is a vector bundle $q : A \to M$ with a bundle map $a : A \to TM$, and an alternating, bilinear form $[-,-] : \Gamma A \times \Gamma A \to \Gamma A$ that satisfies the Jacobi identity (where ΓA is the space of sections of q), such that for any $X, Y \in \Gamma A$ and $u \in C^\infty(M)$:*

- $[X, uY] = u[X, Y] + \big(a(X)(u)\big)Y$
- $a([X, Y]) = [a(X), a(Y)]$ *(i.e. the commutator of the vector fields/differential operators $a(X)$ and $a(Y)$)*

(A morphism of Lie algebroids is a vector bundle map that agrees with $[-,-]$). Note: the last two conditions are trivial for the case $M = \star$ of a Lie algebra since $TM = \star$ is terminal. They give the link to differential geometry on M by allowing X to act as a tangent vector - i.e. a differential operator that can act on Y.)

Theorem 0.1.8. *For any Lie groupoid $\mathcal{G}$, there is an associated Lie algebroid $A\mathcal{G}$ given by the pullback*

(7)
$$\begin{tikzcd} A\mathcal{G} \arrow[r] \arrow[d] & T^sG \arrow[d, "\pi"] \\ M \arrow[r, "i"'] & G \end{tikzcd}$$

of the vertical tangent bundle T^sG to the bundle $s : G \to M$ (i.e. fibrewise tangent bundle). The bracket is also pulled back from the commutator of (vertical) vector fields on T^sG.

In the case where $\mathcal{G}$ is a Lie group, this gives the usual construction of its Lie algebra.

There is also an analog for compact Lie groupoids of Haar measure in the case of a Lie group:

Definition 0.1.9. *A **left Haar system** for a (loc.cpt.) groupoid $\mathcal{G}$ is a family $\{\lambda^x\}_{x \in M}$, where λ^x is a (positive, regular, Borel) measure on $G^x = s^{-1}(x)$.*

Unlike for Haar measure on a Lie group, a (left) Haar system λ^x is not uniquely defined. It is only unique up to a (quasi-invariant, i.e. equivariant) measure μ on M. Typically in differential geometry, measures come from integrating against volume forms. Weinstein used this to define a "volume form on a differentiable stack", where "differentiable stack" here is used to mean "Morita equivalence class of smooth groupoids".

Definition 0.1.10. *If $\mathcal{G}$ is a compact Lie groupoid (i.e. M and fibres of s are all compact) with Lie algebroid A. If $\alpha \in \Gamma(\Lambda^{max} A^*)$ is a nowhere vanishing section of top-dimensional A-forms, and $\beta \in \Lambda^{max} T^* M$. Then the volume of $\mathcal{G}$ is:*

$$vol_{(\alpha,\beta)}(\mathcal{G}) := \int_{x \in M} \Big(\int_{s^{-1}(x)} \alpha_x \Big)^{-1} \beta \tag{8}$$

This naturally extends to a volume form on $\mathcal{G}$ giving volumes of (good) subgroupoids.

By suitably defining how to transport (α, β) over a Morita equivalence, this can be extended to an invariant of the differentiable stack represented by $\mathcal{G}$. It can be used to define *integration over stacks.* Note: defining lower-dimensional forms is less obvious - an equivalent of Stokes' theorem as yet isn't known for differentiable stacks. We return to stacks after the next section.

0.2. **Groupoids and C^*-Algebras.** One way people study groupoids is by their associated C^*-algebras:

Definition 0.2.1. *A C^***-Algebra** is a complex Banach (i.e. complete normed) algebra A with an involution map $\star : A \rightarrow A$ such that $|x^\star x| = |x|^2$.*

These are used in noncommutative geometry because the category of commutative C^*-algebras is equivalent to $\mathbf{Top}^{\mathrm{op}}$.

Example 0.2.2. Two examples show how to represent a space, and a "space-with-symmetry" (think of a G-set for a group G):

- If X is a (locally compact) topological space (manifold), the algebra $C(X)$ of continuous functions ($C^\infty(X)$ of smooth functions) is a C^*-algebra with trivial $\star = \mathrm{Id}$.
- If $\mathcal{G}$ is a (locally compact) topological (or smooth) groupoid, its **groupoid C^*-algebra**, $C^*(\mathcal{G})$ is defined on the space $C(G)$ (or $C^\infty(G)$) of continuous (smooth) functions on morphisms, with the convolution product:

$$(\phi * \psi)(g) = \int \phi(gh)\psi(h^{-1}) d\lambda^{s(g)}(h) \tag{9}$$

  and the involution

$$\phi^\star(g) = \phi(g^{-1}) \tag{10}$$

In the case $\mathcal{G}$ is a group, this is the usual group algebra. In the case $\mathcal{G}$ has only identity morphisms ($G \cong M$), this is just $C(M)/C^\infty(M)$.

Once again, in NCG one typically considers C^*-algebras up to Morita equivalence:

Definition 0.2.3. *The C^*-algebras A and B are **Morita equivalent** if the module categories $Mod(A)$ and $Mod(B)$ are equivalent (i.e. there are functors $F : Mod(A) \rightarrow Mod(B)$ and $G : Mod(B) \rightarrow Mod(A)$ with $G \circ F \simeq \mathrm{Id}_A$ and $F \circ G \simeq \mathrm{Id}_B$).*

Proposition 0.2.4. *A Morita equivalence amounts to A−B-bimodule X and B−A-bimodule Y with $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$, by defining $F(M) = M \otimes_A X$ and $G(N) = Y \otimes_B N$.*

Theorem 0.2.5. *Groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent (as groupoids) iff the C^*-algebras $C(\mathcal{G})$ and $C(\mathcal{G}')$ are Morita equivalent as algebras.*

In fact, the bimodules are just $C(\mathcal{X})$ given $C(A)$- and $C(B)$-module structure in the natural way, where $\mathcal{X}$ is the central object in the Morita equivalence (6).

Example 0.2.6. A common example from NCG is the construction of the (generally noncommutative) C^*-algebra associated to a quotient by a group action. Given a group action $a : G \times X \to X$, this is $C(X /\!\!/ G)$ (or $C^\infty(X /\!\!/ G)$).

A well-known case of this is the "noncommutative torus", $C(\mathbb{T}^2 /\!\!/ \mathbb{R}$, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the usual torus, and $\mathbb{R}$ acts on $\mathbb{T}^2$ by $t : [(a,b)] \mapsto [(a+t, b+kt)]$ for some irrational k. The orbits are dense, so the quotient $\mathbb{T}^2/\mathbb{R}$ is pathological (not Hausdorff, etc.), whereas the C^*-algebra is quite well-behaved.

0.3. **Groupoids and Stacks.** This section is more vague because I find stacks a bit intimidatingly technical at first. Instead, I'll try to give a general idea of the link to groupoids with an intuitive presentation following [No2].

Logically, topological stacks (see Noohi [No1]) underly smooth/analytic/algebraic stacks, but were developed later. Algebraic stacks (Grothendieck, Deligne, Mumford, Artin) came first. The idea was to "categorify" sheaf theory by replacing **Set** with **Gpd**. Thus:

- A **sheaf** on S is a functor $F : S \to \mathbf{Set}^{\mathrm{op}}$, satisfying some descent ("gluing") conditions
- A **stack** on S is a (pseudo)functor $F : S \to \mathbf{Gpd}^{\mathrm{op}}$, satisfying some descent conditions

Where S is a *site* (a category which defines when morphisms into an object $s \in S$ are open covers). The point is that a sheaf, or a stack, on S describes an entity which is "probed by" objects of S.

Example 0.3.1. Some familar examples occur when $S = T(X)$ is the topology of a topological space X seen as a category. That is, the objects are open sets $U \subset X$, and the morphisms are the inclusions. Open covers are defined as usual.

Then a sheaf $F : T(X) \to \mathbf{Set}^{\mathrm{op}}$ determines a set $F(U) \in \mathbf{Set}$ for each open U, and a "restriction map" $F(i) : F(U) \to F(V)$ for each subset $V \subset U$. A stack $F : T(X) \to \mathbf{Gpd}^{\mathrm{op}}$ likewise determines a groupoid $F(U)$ for each open U and restriction map $F(i)$. For example:

- The sheaf of smooth functions $F = [-, \mathbb{R}]$, which gives the set of continuous real-valued functions on each set, and the map giving the restriction to a subset for each inclusion.
- The stack of principal G-bundles, giving: on an open set U the groupoid $F(U)$ whose objects are principal G-bundles on U and whose morphisms are G-bundle maps; for an inclusion $V \subset U$ of open sets, the restriction map $F(i)$ of bundles on U to bundles on V

The descent conditions for both sheaves and stacks generalize the fact that sections (i.e. choices of functions on different U_i) which match on the overlap of the U_i can be glued to give sections on the union. (Though the meaning of "match" is weaker for groupoids than for sets, hence the descent condition is more complicated.)

This sheaf and this stack are both *representable*: there is a space C for which $F = [-, C]$ (so C "represents" F). In the first case, $C = \mathbb{R}$, and $[-, \mathbb{R}]$ denotes the set of functions into $\mathbb{R}$. In the second case, a G-bundle on U amounts to a map from U into the classifying space BG, and $F = [-, BG]$ now denotes a *groupoid* of such maps.

Note that $T(X)$ is a subcategory of **Top**, since every object in $T(X)$ is a topological space and all its maps are continuous. So instead of picking a space X, we make the (slightly technical) definition:

Definition 0.3.2. *A topological stack $\mathcal{X}$ is a stack on $S = \mathbf{Top}$. (For differentiable stack, $S = \mathbf{Diff}$, and so on). It is required satisfy some conditions:*

- $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is representable*
- *there is a topological space X and an epi $p : X \to \mathcal{X}$ representable by (some class of nice maps)*

(where the class of nice maps - "local fibrations" - can be chosen in various ways, all satsifying some axioms, to get slightly different variants of the definition).

(The maps between stacks all rely on the fact that $\mathcal{X}$ can be regarded as a *category fibred in groupoids* - that is, a category over S, where the fibre at U is $F(U)$.)

The main point is, $\mathcal{X}$ is now a stack on **Top**. (In particular, **Top** contains as subcategories all the $T(X)$ for spaces X, so you can restrict attention to one of these if you like.) As for the conditions, the first says the diagonal map is nice; the second asserts there is a topological space with a nice map into $\mathcal{X}$.

Example 0.3.3. Any groupoid $\mathcal{G}$ has a corresponding stack $F : \mathbf{Top} \to \mathbf{Gpd}^{\mathrm{op}}$ given by $F = [-, \mathcal{G}]$ (in particular, spaces are taken as groupoids with $G = M$). Note that this determines $\mathcal{G}$ since $F(\star) = \mathcal{G}$. Having the groupoids of all other maps into $\mathcal{G}$ determines the topology of $\mathcal{G}$.

(Aside: This representability of spaces by stacks is an analog of the Yoneda embedding.)

This last example suggests how a space X can be associated to any stack $\mathcal{X}$ (though X may not determine $\mathcal{X}$). In fact:

Theorem 0.3.4. *Any topological stack $\mathcal{X}$ has an associated topological space X called the* ***coarse moduli space****, with a map $\pi_{mod} : \mathcal{X} \to X$.*

The coarse moduli space X by itself does not determine $\mathcal{X}$. The extra data to do so consists of **inertia groups** I_x at each point $x \in X$ (essentially: self-symmetries of points). But a space with groups associated to each point is precisely a skeletal groupoid (with $M = X$), which determines a Morita equivalence class of groupoids.

The connection to groupoids is further suggested by:

Theorem 0.3.5. *Given any group action $a : G \times X \to X$, there is a* ***quotient stack****, denoted $[X/G]$. The coarse moduli space is just the quotient space X/G.*

References

[BX] K. Behrend and P. Xu. *Differentiable Stacks and Gerbes*. 2006. `arXiv:math/0605694`.

[PH] P.J. Higgins. *Categories and Groupoids*. Van Nostrand Reinhold Mathematical Studies vol 32. 1971.

[KM] K.C.H. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*. London Mathematical Society Lecture Note Series vol 213. Cambridge University Press, 2005.

[IM] I. Moerdijk. *Introduction to the Language of Stacks and Gerbes*. 2002. `arXiv:math/0212266`.

[No1] B. Noohi. *Foundations of Topological Stacks I*. 2005. `arXiv:math/0503247`.

[No2] B. Noohi. *What is a Topological Stack?*. 2005.

[AP] A.L.T. Paterson. *Groupoids, Inverse Semigroups, and their Operator Algebras*. Progress in Mathematics vol 170. Birkhauser. 1998.

[AGP] A. Ramsay, J. Reneault eds. *Groupoids in Analysis, Geometry, and Physics*. Contemporary Mathematics vol 2. American Mathematical Society. 2001.

[JR] J. Reneault. *A Groupoid Approach to C^*-Algebras*. Springer Lecture Notes in Mathematics vol 793. Springer Verlag, 1971.

[AW] A. Weinstein. *The Volume of a Differentiable Stack*. 2008. `arXiv:0809.2130`.

Mathematics Department, University of Western Ontario

E-mail address: `jeffrey.c.morton@gmail.com`