Background

My current work is part of an effort by many researchers to apply ideas from higher dimensional category theory to mathematical physics, and especially to foundational questions in quantum mechanics and quantum field theory. In particular, this work explores the “categorification” of mathematical structures which appear in descriptions of physical systems. This is a broad term which refers to the extension of set-based structures to ones based on categories, or even higher-dimensional categories.

This is related to two fruitful themes in recent mathematical research. The first is that of “Extended” topological quantum field theories (TQFT), as illustrated by the work Lurie on infinity-categories and the classification of such TQFTs [11], and the program of Freed, Hopkins, Lurie, and Teleman, for describing fully local field theories derived from gauge theories [6]. This “extension” of field theories is closely connected with a view of categorification known as delooping, in which a given category is found to be embedded naturally in a higher-dimensional category. In particular, classification of TQFTs involves the delooping of the monoidal category of manifolds and cobordisms.

A further part of this theme involves the premise that such extended field theories are most naturally based on “higher” gauge theory. This represents fields in terms of connections on gerbes, which are higher-categorical analogs for principal bundles. The role of the structure group for a bundle is then taken by a higher-categorical object. This aspect of a categorical program for field theories has been developed Picken et al. [20, 12], and extended to infinity-categories by Schreiber and others [5, 2].

The second major theme is in representation theory, in which the combinatorial structure of certain algebras is shown to arise as a easily-manipulated monoidal categories [13, 10]. This idea was motivated by, and has been rather successfully extended to the representation theory of these algebras, and their role in geometric invariants (for instance, [23]). One motivating idea behind this program has been to find analogs of Reshitikhin-Turaev-type invariants of 3-manifolds in the context of 4-manifolds, treating categorification as a raising of dimension, as in Khovanov’s categorification of the Jones polynomial to a homology theory [9].

Both of these themes have played an important role in my research to date, connected by the ”groupoidification” program of Baez and Dolan [4]. This program involves constructions in a category *Span(Gpd)* whose objects are groupoids and whose morphisms are certain diagrams called spans. This category has many desirable features, being a “dagger-abelian, dagger-monoidal” category, which is a setting in which one can reason about quantum mechanical structures, and which has many applications to both quantum physics and quantum computation [1, 7].

This models quantum systems and processes using a representation of *Span(Gpd)* on *FinHilb*, the category of vector spaces and linear maps. ”Groupoidifications” of a structure in *Vect* are structures in *Span(Gpd)* whose image is the specified one in *Vect*. This has been done is several cases, including geometric representation theory.

Applications of groupoidification to physics regard groupoids as representing classical configurations of systems, together with local symmetry information. Processes from systems described by groupoids A to B are described by spans: these are pairs of maps from X, the groupoid of histories, pick out starting and ending configurations in the groupoids A and B. Some applications of this program to topology and physics arise from the fact that groupoids can be used to representing spaces (of classical *states*) and their

Background Continued

local *symmetries*. The groupoids are then ”fine moduli spaces” for structures. Processes are described by *spans* of groupoids: that is, pairs of maps A ← X → B, where X is the space of histories starting in the space A of states and ending in B.

Work to Date

My work to date has used the methods described above to examine some toy models of quantum physical systems: the quantum harmonic oscillator, and topological field theories.

I described, following Baez and Dolan, a groupoidification of the Fock space and algebra of observables for the quantum harmonic oscillator [15]. This is a fundamental entity in quantum field theory, since any field can be treated as a system of interacting oscillators. Its groupoidification gives a direct combinatiorial interpretation of the calculus of Feynman diagrams used in calculating probablities in this system. With collaborator Jamie Vicary, I described [19] a higher-categorical refinement of this model, which provides a combinatorial interpretation of Khovanov’s categorification of the Heisenberg algebra (the algebra associated to the harmonic oscillator).

Khovanov’s diagrammatic category has a natural model in terms of bimodules for the groupoid algebra of the symmetric groupoid. This groupoid is in turn the natural result of a 2-categorical analog of the “Fock monad” described by Vicary, which gives the Fock space for a Hilbert space. This is the “free symmetric monoidal category” monad. This is a particularly interesting case for groupoidifictation. We also described how the methods used there can give combinatorial groupoidifications of certain other categorified algebras developed by Khovanov and Lauda [10].

I extended the groupoidification functor [17] to a construction called ”2-linearization”, which builds on a higher category called $Span_2(Gpd)$, which contains the original category of spans and groupoids. $Span_2(Gpd)$ is a symmetric monoidal bicategory with duals and an abelian structure, it provides a higher-categorical analog which generalize the dagger-abelian, dagger-monoidal categories, which generalize the properties of the category of Hilbert spaces, used in quantum mechanical models described by Abramsky, Coecke et. al. and has higher- categorical analogs of the properties of the original [21, 8]. The 2-categorical form of this construction should rely on analogous structures for the monoidal 2-category of groupoids and spans on which 2-linearization acts.

Both groupoidification and 2-linearization can be interpreted in terms of a “sum- over-histories” achived through two adjoint operations. These are ”pull-back” from A to X, and ”push-forward” to B, of functions and representations, respectively. Specifically, 2-linearization assigns a 2-vector space to that system, which is the representation category of a groupoid representing the moduli stack (which is defined up to equivalence). This can be understood as describing the superselection structure of a quantum theory. The pull-back and push-forward operations are then the restriction and induction functors for representations. These give 2-linear maps between representation categories characterize systems with boundary, represented by spans of groupoids, and decompose according to the superselection sectors for the boundary conditions. Natural transformations then describe time evolution of these systems. When the boundary is empty, this will reproduce the time evolution map defined by groupoidification.

I have recently extended the 2-linearization process to allow for ”twisting” by data which arises from group cohomology, and which reproduces the Dijkgraaf-Witten model for manifolds with boundary in this framework [18]. This is a continuation of work in my Ph.D. thesis [16], which addresses the program for extended field theory, in the context of the

Work to Date (Continued)

Chern-Simons theory and (2+1)-dimensional quantum gravity. In that setting, the categorification, or "extension" of the field theory permits coupling of the field to inserted particles.

Research Plan

My current research projects aim to extend the work discussed above in several ways. First, I aim to improve the tools for use in studying more mathematically and physically interesting systems. Second, for each such area of improvement, there are concrete examples which can be constructed, some of which have direct interest, such as manifold invariants, or broader classes of field theories. So developing categorified forms of these examples will be one avenue of research. Finally, in each area of improvement, there are novel abstract principles which appear, and can be explored.

One project I have recently begun a project which is motivated directly by a class of examples is an attempt to groupoidify q-deformations of the categorified algebras which can be given in $Span_2(Gpd)$, following work on q-deformed groupoidification in terms of vector spaces over finite fields, by Baez, Hoffnung et al.

This should further develop the correspondence between the groupoidification program and Khovanov-Lauda type categorification of Hopf algebras, and in particular quantum groups, in terms of 2-categories presented by diagrammatic means. These can often be given representations in terms of 2-categories whose morphisms are bimodules. We expect that this will be related to the 2-linearization construction, since this relies on the adjunction between induction and restriction functors for representations. These can be represented by exactly the bimodules used in Khovanov-type categorifications. These categorifications have recently begun to play an important role in building topological invariants in the form of Khovanov homology and other related constructions.

One existing project, with Jamie Vicary, relates to developing higher-categorical versions of the well-developed work in dagger-abelian, dagger-monoidal categories as a setting for quantum physics and computation. In our categorification of the Heisenberg algebra, we use a 2-categorical analog of the "Fock monad" of Vicary. This is an algebraic characterization, in a general category, of the construction of multiparticle systems from single-particle systems. By developing the theory, we show how our groupoidification can be generalized far beyond the Heisenberg algebra, for example to give analogous Fock spaces for extended field theories with groupoidifications as above.

One path for extending my program which bears directly on topological invariants is the use of 2-linearization to construct topological quantum field theories. With collaborator Derek Wise, I aim to give a description of more physically realistic topological gauge theories and especially ETQFT from compact Lie groups. This entails developing an extension of 2-linearization to the setting of measured groupoids, higher groupoids.

This project aims to construct Chern-Simons theory (related to quantum gravity in 2+1 dimensions), and other field theories based on compact Lie groups, in the 2-categorical setting. This is most immediately presented in terms of a description of 2-Hilbert spaces given by Baez, Baratin, Freidel and Wise [3] in terms of von Neumann algebras, and a related characterization of functors in terms of bimodules. The general description will involve characterizing 2-Hilbert spaces in terms of categories of equivariant measurable sheaves of Hilbert spaces on measurable groupoids. These take the place of representation categories of finite groupoids, and the induction makes use of the theory of integration on measured groupoids. Since these measured groupoids arise from Lie

Research Plan (Continued)

groupoids using Haar measure for the gauge group *G* and related constructions, this can also be described in terms of Weinstein's formulas for the volume of a differentiable stack [24]. These will the allow the 2-linearization construction, and therefore the construction of extended TQFTs, to be extended to Lie groups.

A different aspect of my research program is to use *n*-Hilbert space representations for *n*-fold spans of *n*-groupoids. This would explicitly capture the geometric construction of Freed-Hopkins-Lurie-Teleman, extending TQFTs on *n*-dimensional manifolds "down to a point" [6]. The project is to analyze the algebraic structure of the span categories underlying this, and prove results about adjointness of their higher morphisms. This would allow the representation *n*-categories of *n*-groupoids to play the role of Hilbert spaces in a multiply categorified analog of the Baez-Dolan program.

To this end, I have been collaborating with Roger Picken, who, along with Joao Martins, has studied the geometry connections in "Higher Gauge Theory". This is a higher- categorical analog of ordinary gauge theory, based on a "2-group", a higher-categorical structure which takes the place of the gauge group [2]. They relate to the theory of connections on gerbes, and give holonomies for higher-dimensional surfaces than paths within a manifold. These holonomies take value in 2-groups, often described in terms of crossed modules of groups. There has been some interest in higher gauge theories, such as BFCG-theory, in the quantum gravity community [14], and also in string theory, since these fields naturally couple to extended objects such as strings. In collaboration with Picken, I have been working to give a description of the moduli spaces of higher gauge theories as n-groupoids (which present n-stacks). This led to a further general study of 2-group symmetries. We then show that these occur naturally in higher gauge theories, but in a way which is not immediately evident in standard presentations, by using higher structures analogous to transformation groupoids.

This should lead to a generalization of the theory of 2-linearization to even higher categories, and therefore further-extended field theories in higher codimension. We aim to develop a program of 2-groupoidification for TQFT based on higher gauge theory, by considering the appropriate adjunction between induction and restriction of 2-group representations. This will allow construction of extended TQFT in higher dimension, giving, for instance, theories on 4-manifolds based on 2-groups. The cohomological twisting of 2-linearization suggests the possibility to develop a connection between such theories and the correspondence between field theories and cohomology theories of the Stolz-Teichner program [22].

This is motivated by examples of higher gauge theories such as BF theory, which is a limit of (3+1)-dimensional quantum gravity, with boundaries that correspond to moving particles and strings in the background of this theory. The goal is that this framework will provide a higher-algebraic context and tools for studying this situation. It is also, therefore, related to invariants of 4-manifolds based on gauge theory.

The guiding theme behind all these generalizations is the confluence, in the world of topology and manifold invariants, of ideas in higher category theory, and simple models in mathematical physics. This creates an exciting opportunity for interaction and exchange of ideas between these fields.

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