Higher Gauge Theory and 2-Group Actions

(Joint work with Roger Picken)

Jeffrey C. Morton

University of Hamburg

Brno, Czech Republic
Dec 13, 2012

Outline

- Motivation
- Groupoids of Connections and Gauge Transformations
- 2-Groups
- 2-Groupoids of Higher Connections
- Actions of 2-Groups
- Transformation 2-Groupoid

Motivation

- Generalize Gauge Theory to Higher Gauge Theory
- Topological Field Theory
  - Geometric Invariants
  - Homotopy QFT/Sigma Models
  - Maps into target space X
  - X as Classifying Space for n-group(oid)
- Generalizing Symmetry
  - Symmetry of Moduli Space
  - From Group Actions to 2-Group Actions

Groupoids of Connections

Definition

A group G is a one-object category whose morphisms are all invertible.

Definition

The **fundamental groupoid** $\Pi_1(M)$ of a manifold M has:

- Objects: Points of M
- Morphism: $Hom(x,y)$ - homotopy classes of paths in M from x to y

Definition

A **flat** G**-connection** is a functor

$$A : \Pi_1(M) \to G$$

which assigns *holonomies* to paths in M.

Definition

A **gauge transformation** $\alpha : A \to A'$ is a natural transformation of functors so that $\alpha_x \in G$ satisfies $\alpha_y A(\gamma) = A'(\gamma)\alpha_x$ for each path $\gamma : x \to y$.

Flat connections and natural transformations form the objects and morphisms of the *groupoid of flat connections*

$$\mathcal{A}_0(M) = Fun(\Pi_1(M), G)$$

Note: this is equivalent as a category (see Schreiber-Waldorf) to the more usual definition in terms of flat bundles with connection, usually described in terms of a field of 1-forms.

Example

The groupoid of flat G-connections on the circle S^1

- $\Pi_1(S^1) \simeq \mathbb{Z}$ (as a one-object category)
- $g : \mathbb{Z} \to G$ is determined by $g = g(1) \in G$. (The *holonomy around the circle*).
- A natural transformation is a conjugacy relation: $\gamma : g \to g'$ assigns $\gamma \in G$ to the object of $\mathbb{Z}$
- Naturality says that $g'h = hg$, or simply $g' = hgh^{-1}$. (It acts by *conjugation at a point*).

Definition

A group action ϕ on a set X is a functor

$$\phi : G \rightarrow \textbf{Sets}$$

where $X = \phi(\star)$ is the image of the unique object of G. Equivalently

Definition

The **transformation groupoid** of an action of a group G on a set X is the groupoid $X /\!\!/ G$ with:

- **Objects**: All $x \in X$ (really pairs $(x, \star)$)
- **Morphisms**: Pairs (x, g), where $s(x, g) = x$, and $t(x, g) = gx$
- **Composition**: $(gx, g') \circ (x, g) = (x, g'g)$

Proposition

The groupoid of flat connections, $\mathcal{A}_0(S^1)$ is equivalent to the transformation groupoid $G /\!/ G$ of the adjoint action of the group G on itself.

This is a special case of the more general fact:

Proposition

If M is a connected manifold, $\mathcal{A}_0(M)$ is equivalent to the transformation groupoid of an action of the group of all gauge transformations on the space of all connections.

This is the statement we want to generalize to 2-groups. (We will prove a slightly different version of it for technical reasons.)

2-Groups and Crossed Modules

Definition

A **2-group** $\mathcal{G}$ is a 2-category with one object, and all morphisms and 2-morphisms invertible.

The 2-category of 2-groups is equivalent to the 2-category of **crossed modules**.

Definition

A crossed module consists of $(G, H, \rhd, \partial)$, where G and H are groups, $G \rhd H$ is an action of G on H by automorphisms and $\partial : H \to G$ a homomorphism, satisfying the equations:

$$\partial(g \rhd h) = g\partial(h)g^{-1}$$

and

$$\partial h \rhd h' = hh'h^{-1}$$

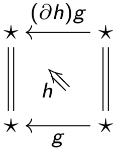
Definition (Part 1)

The 2-group given by $(G, H, \rhd, \partial)$ has:

- **Objects**: Elements of G

$$\star \xleftarrow[g]{} \star$$

- **Morphisms**: Pairs (g, h),

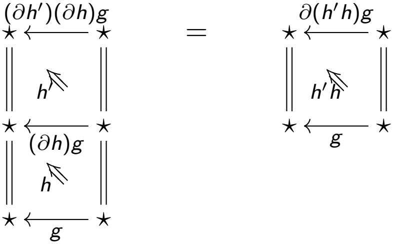


(source and target maps $s(g,h) = g$ and $t(g,h) = (\partial h)g$ as shown).

Definition (Part 2)

Vertical Composition:

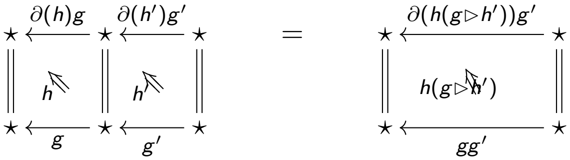
$$((\partial h)g, h') \circ (g, h) = (g, h'h).$$



Definition (Part 3)

Horizontal Composition:
By multiplication in the group $G \ltimes H$:

$$(g,h)(g',h') = (gg', h(g \rhd h'))$$



Note that properties of crossed products mean that

$$\partial(h(g \rhd h'))g' = (\partial h)g(\partial h')g'$$

Actions of 2-Groups on Categories

By analogy with groups, the most natural definition of 2-group actions is in terms of 2-functors:

Definition

A 2-group $\mathcal{G}$ acts (strictly) on a category $\mathbf{C}$ if there is a (strict) 2-functor:

$$\Phi : \mathcal{G} \rightarrow \mathbf{Cat}$$

whose image lies in $End(\mathbf{C})$.

So then:

- $\Phi(*) = \mathbf{C}$
- $\gamma \in Mor(\mathcal{G})$ gives an endofunctor:

$$\Phi_\gamma : \mathbf{C} \rightarrow \mathbf{C}$$

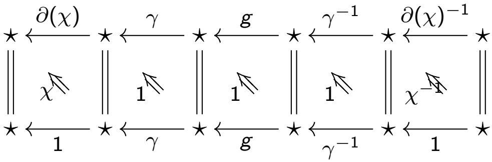
- $(\gamma, \chi) \in 2Mor(\mathcal{G})$ gives a natural transformation:

$$\Phi_{(\gamma,\chi)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\chi)\gamma}$$

Adjoint Action of a 2-Group

The adjoint action of a 2-group $\mathcal{G}$ treats $\mathcal{G}$ as both a 2-group acting, and a (monoidal) category being acted on, whose objects are the morphisms of the 2-group $\mathcal{G}$.

For $\mathcal{G} = (G, H, \rhd, \partial)$, the action of 1-morphisms in G on 1-morphisms is exactly conjugation in G. For 2-morphisms, the following diagram shows how it should work:



Note: this diagram illustrates that inverses of 2-morphisms have the slightly awkward labelling:

$$(\gamma, \chi)^{-1} = (\gamma^{-1}, (\gamma^{-1} \rhd \chi^{-1}))$$

Definition (Adjoint Action of a 2-Group - Part 1)

Suppose $\mathcal{G}$ is the 2-group given by a crossed module $(G, H, \rhd, \partial)$. Then define a functor:

$$\Phi : \mathcal{G} \to \mathbf{Cat}$$

with image in $End(\mathcal{G})$ in the following way. For each object $\gamma \in Ob(\mathcal{G})$, the endofunctor

$$\Phi_\gamma : \mathcal{G} \to \mathcal{G}$$

has the object map:

$$\Phi_\gamma(g) = \gamma g \gamma^{-1}$$

and the morphism map:

$$\Phi_\gamma(g, \chi) = (\gamma g \gamma^{-1}, \gamma \rhd \chi)$$

Lemma

The object map gives a (monoidal) endofunctor $\Phi_\gamma : \mathcal{G} \to \mathcal{G}$.

Definition (Adjoint Action - Part 2)

For each 2-morphism $(\gamma, \chi) \in \mathcal{G}$ there is a natural transformation:

$$\Phi_{(\gamma,\chi)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\chi)\gamma}$$

It is given, at a given object g, by:

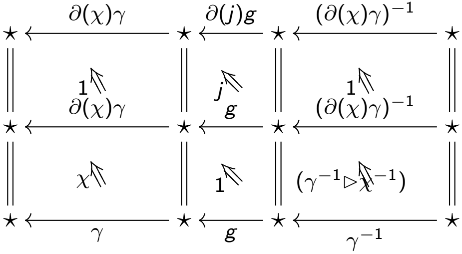
$$\Phi_{(\gamma,\chi)}(g) = (\gamma g \gamma^{-1}, \chi(\gamma g \gamma^{-1}) \rhd \chi^{-1}))$$

(That is, "conjugation by (γ, χ)".)

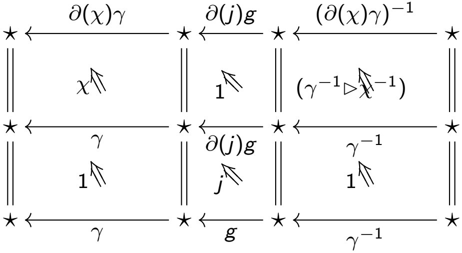
Lemma

The transformation this defines is natural.

The proof that $\Phi_{(\gamma,\chi)}$ is natural amounts to the equality of:



and



Higher Gauge Theory

Goal: Use 2-groups to generalize preceding constructions of connections and gauge transformations.

Definition

The **fundamental 2-groupoid** of a manifold M has:

- Objects: Points of M
- Morphisms: $Hom(x, y)$ - paths in M from x to y
- 2-Morphisms: $Hom(p_1, p_2)$ - homotopy classes of homotopies of paths from p_1 to p_2

2-Groupoid of Connections

Definition

The *gauge 2-groupoid* for a 2-group $\mathcal{G}$ on a manifold M is:

$$\mathcal{A}_0(M, \mathcal{G}) = Hom(\Pi_2(M), \mathcal{G})$$

This is the 2-functor 2-category, which has:

- Objects: 2-Functors from $\Pi_2(M)$ to $\mathcal{G}$, called **Connections**
- Morphisms: Natural transformations between functors, called **Gauge Transformations**
- 2-Morphisms: Modifications between natural transformations, called **Gauge Modifications**

(The term "gauge modification" appears not to be in common use yet!)

Category of 2-Group Connections

The following applies to a manifold M with a decomposition into cells, with vertex set V, edge set E, and face set F.

Definition (Category of Connections - Part 1)

The **category of connections**, $\mathbf{Conn} = \mathbf{Conn}(\mathcal{G}, (V, E, F))$, has the following:

- Objects of **Conn** consist of pairs of the form

$$\{(g,h) | g : E \rightarrow G,\ h : F \rightarrow H \text{ s.t. } \prod_{e \in \partial f} g(e) = \partial h(f)\}$$

- **Morphisms**: Morphisms of **Conn** with a given source (g, h) are labelled by $\eta : E \rightarrow H$.

Definition (Category of Connections - Part 2)

The target of a morphism from (g, h) labelled by η is (g', h') with:

$$g'(e) = \partial(\eta(e))g(e)$$

and

$$h'(f) = h(f)\hat{\eta}(\partial(f))$$

The term $\hat{\eta}$ is the total H-holonomy around the boundary of the face f, whose edges are e_i (taken in order):

$$\hat{\eta}(\partial(f)) = \prod_{e_j \in \partial(f)} (\prod_{i=1}^{j} g_i) \rhd \eta_j$$

Note: the morphisms of **Conn** include *part* of what are usually called "gauge transformations" of 2-group connections in higher gauge theory, but not *all* of them!

We define a 2-group which acts on **Conn** to discover the rest... and all "gauge modifications", which do not occur in normal gauge theory!

2-Group of Gauge Transformations

Definition

Given M with cell decomposition including (V, E, F) as above, the **2-group of gauge transformations** is **Gauge** $= \mathcal{G}^V$, which has:

- objects $\gamma : V \to G$
- morphisms (γ, χ) with $\chi : V \to H$
- 2-group structure given by ∂ and $\triangleright$ acting pointwise as in $\mathcal{G}$

Claim: there is a natural action of **Gauge** on **Conn**:

$$\Phi : \mathbf{Gauge} \to End(\mathbf{Conn})$$

Action of **Gauge** on **Conn**

Definition (Gauge 2-Group Action - Part 1)

The action of **Gauge** on **Conn** is given by:

- An object $\gamma : V \to G$ of **Gauge** gives a functor $\Phi(\gamma) : \mathbf{Conn} \to \mathbf{Conn}$, "conjugation by γ" acting:
  - on objects $(g, h) \in$ **Conn** by:

    $$\Phi(\gamma)(g,h) = (\hat{g}, \gamma \rhd h)$$

    where

    $$\hat{g}(e) = \gamma(s(e))^{-1} g(e) \gamma(t(e))$$

    and

    $$(\gamma \rhd h)(e) = \gamma(s(e_1)) \rhd h(f)$$

  - on morphisms $((g,h),\eta)$ by:

    $$\Phi(\gamma)((g,h),\eta) = ((\hat{g}, \gamma \rhd h), \eta)$$

Definition (Gauge 2-Group Action - Part 2)

- A morphism (γ, χ) of **Gauge** gives a natural transformation

$$\Phi(\gamma, \chi) : \Phi(\gamma) \Rightarrow \Phi(\gamma') : \mathbf{Conn} \to \mathbf{Conn}$$

where $\gamma' = \partial(\chi)\gamma$, defined as follows: for each object $(g, h) \in \mathbf{Conn}$,

$$\Phi(\gamma, \chi)(g, h) = ((\tilde{g}, \tilde{h}), \tilde{\eta})$$

where

- $\tilde{g}(e) = \gamma(s(e))^{-1} g(e) \gamma(t(e))$
- $\tilde{h}(f) = h(f)$
- $\tilde{\eta}(e) = \gamma(s(e))^{-1} \rhd (\chi(s(e))^{-1}.g \rhd \chi(t(e)))$

for each $e \in E$, $f \in F$.

Goal: 2-Group analog of the theorem that $\mathcal{A}_0(M)$ is equivalent to a transformation groupoid, for this action.

Gauge Groupoid as 2-Groupoids

Goal: We want to construct an analog $\mathbf{C}/\!\!/\mathcal{G}$ for a transformation groupoid. It should be a 2-groupoid associated to a 2-group action.

$$\Phi : \mathcal{G} \rightarrow End(\mathbf{C}) = Cat(\mathbf{C}, \mathbf{C})$$

It is easier if we understand $\mathcal{G}$ as a group object in **Cat**, since the action can also be expressed (via "currying"):

$$\triangleright : \mathcal{G} \times \mathbf{C} \rightarrow \mathbf{C}$$

The map $\triangleright$ is an action if it satisfies:

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id} & \mathcal{G} \times \mathbf{C} \\ {\scriptstyle Id \times \triangleright} \downarrow & & \downarrow {\scriptstyle \triangleright} \\ \mathcal{G} \times \mathbf{C} & \xrightarrow[\triangleright]{} & \mathbf{C} \end{array}$$

Idea: consider the construction of $S /\!\!/ G$ for a group action in diagrammatic terms in **Set**, and follow the same construction in **Cat**.

Transformation Double Category

For a group action of G on S, the transformation groupoid is constructed as a pullback in **Set**:

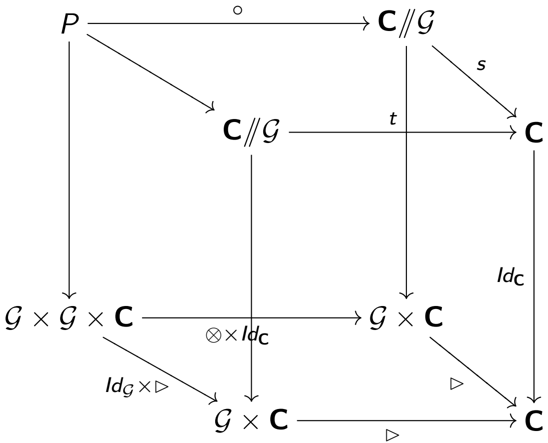
$$\begin{array}{ccc} S /\!\!/ G & \xrightarrow{t} & S \\ \downarrow & & \| \\ G \times S & \xrightarrow[\triangleright]{} & S \end{array}$$

This gives a set of morphisms, whose composition comes from the pullback square (using $s = \pi_2$):

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & S /\!\!/ G \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle s} \\ S /\!\!/ G & \xrightarrow[t]{} & S \end{array}$$

Transformation Double Category

The transformation double category associated to $\triangleright : \mathcal{G} \times \mathbf{C} \rightarrow \mathbf{C}$ is constructed by the analogous pullbacks in **Cat**:



Description of the Transformation Double Category

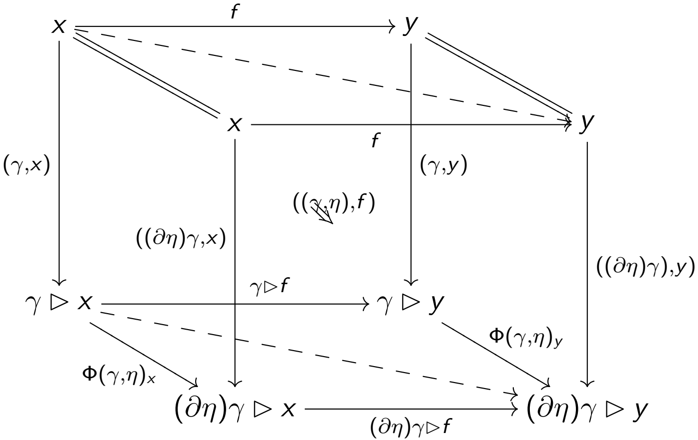
Concretely, $\mathbf{C}/\!\!/\mathcal{G}$ is part of a category internal in **Cat**.
The category of objects is $\mathbf{C}$, with objects and morphisms:

$$x \xrightarrow{f} y$$

The category of morphisms is $\mathbf{C}/\!\!/\mathcal{G}$. Its objects and morphisms are the vertical arrows and squares of:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle (\gamma,x)}\downarrow & ((\gamma,\eta),f) & \downarrow{\scriptstyle ((\partial\eta)\gamma,y)} \\ \gamma \rhd x & \longrightarrow & (\partial\eta)\gamma \rhd y \end{array}$$

The squares are diagonals of the naturality cubes:



(Note: the cube's four side faces are themselves special cases of squares when $f = Id_x$ or $\eta = 1_H$.)

Folding to a 2-Category

Proposition

Claim: For a manifold M (with cell decomposition (V, E, F)), the transformation double category **Conn**$/\!\!/$**Gauge** *is equivalent to the functor 2-category $Hom(\Pi_2(M, (V, E, F)), \mathcal{G})$.*

To parse this, we must "fold" the double category $\mathbf{C}/\!\!/\mathcal{G}$ to give a 2-category $\widehat{\mathbf{C}/\!\!/\mathcal{G}}$ with:

- The same objects as $\mathbf{C}/\!\!/\mathcal{G}$ (hence of $\mathbf{C}$)
- Morphisms: Composites of horizontal and vertical morphisms of $\mathbf{C}/\!\!/\mathcal{G}$, i.e.:
  - morphisms of the object category
  - objects of the morphism category
- 2-Morphisms: squares of $\mathbf{C}/\!\!/\mathcal{G}$

Example 1: Connections on the Circle

Our general claim is that $\mathcal{A}_0(M,(V,E,F)) \cong \widehat{\mathbf{Conn}/\!\!/\mathbf{Gauge}}$. For the circle, we have worked this out in detail already.

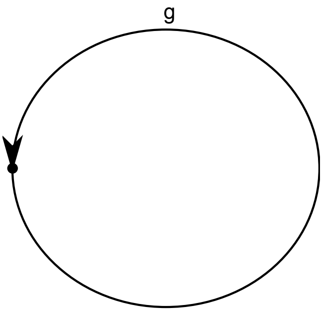
$\mathcal{A}_0(S^1) = Hom(\Pi_2(S^1),\mathcal{G})$, with:

- Objects: Functors $F : \Pi_2(S^1) \rightarrow \mathcal{G}$, which are determined by $F(1) \in G$
- Morphisms: Natural transformations $n : F \Rightarrow F'$ determined by $\gamma \in G$ and $\eta \in H$
- 2-Morphisms: Modifications $\phi : n \Rightarrow n'$ determined by $\chi \in H$

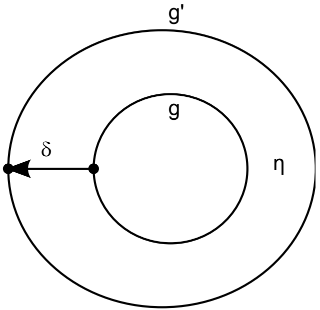
Theorem

There is an equivalence of 2-groupoids $\mathcal{A}_0(S^1) \cong \widehat{\mathcal{G}/\!\!/\mathcal{G}}$.

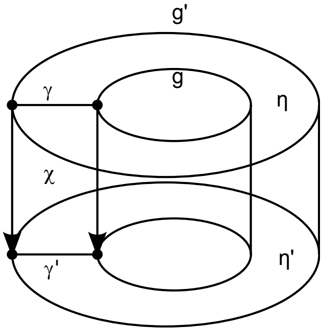
Example 1: Connection on a Circle



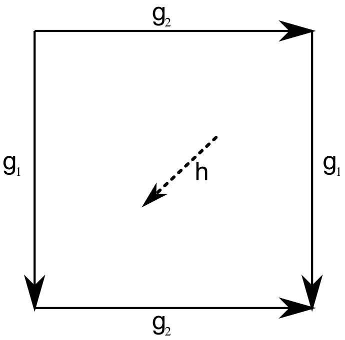
Example 1: Gauge Transformation on a Circle



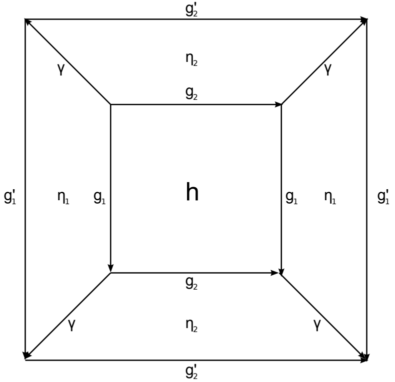
Example 1: Gauge Modification on a Circle



Example 2: Connection on a Torus



Example 2: Gauge Transformation on a Torus



Example 2: Gauge Modification on a Torus