Higher Gauge Theory and 2-Group Actions

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Outline

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- 2-Groups
- 2-Groupoids of Higher Connections
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Motivation

- Generalize Gauge Theory to Higher Gauge Theory
- Topological Field Theory
  - Geometric Invariants
  - Homotopy QFT/Sigma Models
  - Maps into target space $X$
  - $X$ as Classifying Space for $n$-group(oid)
- Generalizing Symmetry
  - Symmetry of Moduli Space
  - From Group Actions to 2-Group Actions
**Groupoids of Connections**

**Definition**
A group $G$ is a one-object category whose morphisms are all invertible.

**Definition**
The **fundamental groupoid** $\Pi_1(M)$ of a manifold $M$ has:
- **Objects**: Points of $M$
- **Morphism**: $\text{Hom}(x, y)$ - homotopy classes of paths in $M$ from $x$ to $y$

**Definition**
A **flat $G$-connection** is a functor

$$A : \Pi_1(M) \to G$$

which assigns **holonomies** to paths in $M$. 
Definition

A **gauge transformation** $\alpha : A \to A'$ is a natural transformation of functors so that $\alpha_x \in G$ satisfies $\alpha_y A(\gamma) = A'(\gamma) \alpha_x$ for each path $\gamma : x \to y$.

Flat connections and natural transformations form the objects and morphisms of the **groupoid of flat connections**

$$A_0(M) = \text{Fun}(\Pi_1(M), G)$$

Note: this is equivalent as a category (see Schreiber-Waldorf) to the more usual definition in terms of flat bundles with connection, usually described in terms of a field of 1-forms.
Example

The groupoid of flat $G$-connections on the circle $S^1$

- $\Pi_1(S^1) \simeq \mathbb{Z}$ (as a one-object category)
- $g : \mathbb{Z} \to G$ is determined by $g = g(1) \in G$. (The holonomy around the circle).
- A natural transformation is a conjugacy relation: $\gamma : g \to g'$ assigns $\gamma \in G$ to the object of $\mathbb{Z}$
- Naturality says that $g'h = hg$, or simply $g' = hgh^{-1}$. (It acts by conjugation at a point).
**Definition**

A group action \( \phi \) on a set \( X \) is a functor

\[
\phi : G \to \text{Sets}
\]

where \( X = \phi(\star) \) is the image of the unique object of \( G \). Equivalently

**Definition**

The **transformation groupoid** of an action of a group \( G \) on a set \( X \) is the groupoid \( X \rightrightarrows G \) with:

- **Objects**: All \( x \in X \) (really pairs \( (x, \star) \))
- **Morphisms**: Pairs \( (x, g) \), where \( s(x, g) = x \), and \( t(x, g) = gx \)
- **Composition**: \( (gx, g') \circ (x, g) = (x, g'g) \)
Proposition

The groupoid of flat connections, $\mathcal{A}_0(S^1)$ is equivalent to the transformation groupoid $G \rtimes G$ of the adjoint action of the group $G$ on itself.

This is a special case of the more general fact:

Proposition

If $M$ is a connected manifold, $\mathcal{A}_0(M)$ is equivalent to the transformation groupoid of an action of the group of all gauge transformations on the space of all connections.

This is the statement we want to generalize to 2-groups. (We will prove a slightly different version of it for technical reasons.)
2-Groups and Crossed Modules

Definition

A 2-group $G$ is a 2-category with one object, and all morphisms and 2-morphisms invertible.

The 2-category of 2-groups is equivalent to the 2-category of crossed modules.

Definition

A crossed module consists of $(G, H, \triangleright, \partial)$, where $G$ and $H$ are groups, $G \triangleright H$ is an action of $G$ on $H$ by automorphisms and $\partial : H \to G$ a homomorphism, satisfying the equations:

$$\partial(g \triangleright h) = g\partial(h)g^{-1}$$

and

$$\partial h \triangleright h' = hh'h^{-1}$$
Definition (Part 1)

The 2-group given by \((G, H, \triangleright, \partial)\) has:

- **Objects**: Elements of \(G\)

  \[
  \star \xleftarrow{g} \star
  \]

- **Morphisms**: Pairs \((g, h)\),

  \[
  \star \xleftarrow{(\partial h)g} \star
  \\
  \star \xleftarrow{h} \star
  \\
  \star \xleftarrow{g} \star
  \\
  \]

  (source and target maps \(s(g, h) = g\) and \(t(g, h) = (\partial h)g\) as shown).
Definition (Part 2)

**Vertical Composition:**

\[ ((\partial h)g, h') \circ (g, h) = (g, h'h). \]
Definition (Part 3)

**Horizontal Composition:**
By multiplication in the group \( G \ltimes H \):

\[
(g, h)(g', h') = (gg', h(g \triangleright h'))
\]

Note that properties of crossed products mean that

\[
\partial(h(g \triangleright h'))g' = (\partial h)g(\partial h')g'
\]
Actions of 2-Groups on Categories

By analogy with groups, the most natural definition of 2-group actions is in terms of 2-functors:

**Definition**

A 2-group \( \mathcal{G} \) acts (strictly) on a category \( \mathbf{C} \) if there is a (strict) 2-functor:

\[
\Phi : \mathcal{G} \to \mathbf{Cat}
\]

whose image lies in \( \text{End}(\mathbf{C}) \).

So then:

- \( \Phi(\ast) = \mathbf{C} \)
- \( \gamma \in \text{Mor}(\mathcal{G}) \) gives an endofunctor:
  \[
  \Phi_\gamma : \mathbf{C} \to \mathbf{C}
  \]
- \( (\gamma, \chi) \in 2\text{Mor}(\mathcal{G}) \) gives a natural transformation:
  \[
  \Phi_{(\gamma,\chi)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\chi)\gamma}
  \]
Adjoint Action of a 2-Group

The adjoint action of a 2-group $G$ treats $G$ as both a 2-group acting, and a (monoidal) category being acted on, whose objects are the morphisms of the 2-group $G$.

For $G = (G, H, \triangleright, \partial)$, the action of 1-morphisms in $G$ on 1-morphisms is exactly conjugation in $G$. For 2-morphisms, the following diagram shows how it should work:

\[
\begin{array}{cccccccc}
\ast & \leftarrow & \partial(\chi) & \leftarrow & \gamma & \leftarrow & g & \leftarrow & \gamma^{-1} & \leftarrow & \partial(\chi)^{-1} & \leftarrow & \ast \\
\ast & \leftarrow & \chi & \leftarrow & 1 & \leftarrow & 1 & \leftarrow & 1 & \leftarrow & \chi^{-1} & \leftarrow & \ast \\
\ast & \leftarrow & 1 & \leftarrow & \gamma & \leftarrow & g & \leftarrow & \gamma^{-1} & \leftarrow & 1 & \leftarrow & \ast \\
\end{array}
\]

Note: this diagram illustrates that inverses of 2-morphisms have the slightly awkward labelling:

\[
(\gamma, \chi)^{-1} = (\gamma^{-1}, (\gamma^{-1} \triangleright \chi^{-1}))
\]
Definition (Adjoint Action of a 2-Group - Part 1)

Suppose $\mathcal{G}$ is the 2-group given by a crossed module $(\mathcal{G}, H, \triangleright, \triangleright)$. Then define a functor:

$$\Phi : \mathcal{G} \to \textbf{Cat}$$

with image in $\text{End}(\mathcal{G})$ in the following way. For each object $\gamma \in \text{Ob}(\mathcal{G})$, the endofunctor

$$\Phi \gamma : \mathcal{G} \to \mathcal{G}$$

has the object map:

$$\Phi \gamma(g) = \gamma g \gamma^{-1}$$

and the morphism map:

$$\Phi \gamma(g, \chi) = (\gamma g \gamma^{-1}, \gamma \triangleright \chi)$$

Lemma

The object map gives a (monoidal) endofunctor $\Phi \gamma : \mathcal{G} \to \mathcal{G}$. 
Definition (Adjoint Action - Part 2)

For each 2-morphism \((\gamma, \chi) \in \mathcal{G}\) there is a natural transformation:

\[ \Phi_{(\gamma, \chi)} : \Phi_\gamma \Rightarrow \Phi_\partial(\chi)\gamma \]

It is given, at a given object \(g\), by:

\[ \Phi_{(\gamma, \chi)}(g) = (\gamma g \gamma^{-1}, \chi(\gamma g \gamma^{-1}) \triangleright \chi^{-1}) \]

(That is, “conjugation by \((\gamma, \chi)\”.)

Lemma

*The transformation this defines is natural.*
The proof that $\Phi_{(\gamma, \chi)}$ is natural amounts to the equality of:

\[ \partial(\chi) \gamma \quad \partial(j)g \quad (\partial(\chi) \gamma)^{-1} \]

\[ \gamma \quad \gamma^{-1} \]

and

\[ \partial(\chi) \gamma \quad \partial(j)g \quad (\partial(\chi) \gamma)^{-1} \]

\[ \gamma \quad \gamma^{-1} \]

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Higher Gauge Theory

Goal: Use 2-groups to generalize preceding constructions of connections and gauge transformations.

Definition

The fundamental 2-groupoid of a manifold $M$ has:

- Objects: Points of $M$
- Morphisms: $\text{Hom}(x, y)$ - paths in $M$ from $x$ to $y$
- 2-Morphisms: $\text{Hom}(p_1, p_2)$ - homotopy classes of homotopies of paths from $p_1$ to $p_2$
Definition

The \textit{gauge 2-groupoid} for a 2-group $G$ on a manifold $M$ is:

$$\mathcal{A}_0(M, G) = \text{Hom}(\Pi_2(M), G)$$

This is the 2-functor 2-category, which has:

- \textbf{Objects}: 2-Functors from $\Pi_2(M)$ to $G$, called \textbf{Connections}
- \textbf{Morphisms}: Natural transformations between functors, called \textbf{Gauge Transformations}
- \textbf{2-Morphisms}: Modifications between natural transformations, called \textbf{Gauge Modifications}

(The term “gauge modification” appears not to be in common use yet!)
Category of 2-Group Connections

The following applies to a manifold $M$ with a decomposition into cells, with vertex set $V$, edge set $E$, and face set $F$.

Definition (Category of Connections - Part 1)

The category of connections, $\text{Conn} = \text{Conn}(\mathcal{G}, (V, E, F))$, has the following:

- **Objects of $\text{Conn}$** consist of pairs of the form
  \[ \{(g, h) | g : E \to G, h : F \to H \text{ s.t. } \prod_{e \in \partial f} g(e) = \partial h(f)\} \]

- **Morphisms**: Morphisms of $\text{Conn}$ with a given source $(g, h)$ are labelled by $\eta : E \to H$. 
Definition (Category of Connections - Part 2)

The target of a morphism from \((g, h)\) labelled by \(\eta\) is \((g', h')\) with:

\[ g'(e) = \partial(\eta(e))g(e) \]

and

\[ h'(f) = h(f)\hat{\eta}(\partial(f)) \]

The term \(\hat{\eta}\) is the total \(H\)-holonomy around the boundary of the face \(f\), whose edges are \(e_i\) (taken in order):

\[ \hat{\eta}(\partial(f)) = \prod_{e_j \in \partial(f)} \left( \prod_{i=1}^{j} g_i \right) \triangleright \eta_j \]
Note: the morphisms of $\text{Conn}$ include part of what are usually called “gauge transformations” of 2-group connections in higher gauge theory, but not all of them!

We define a 2-group which acts on $\text{Conn}$ to discover the rest... and all “gauge modifications”, which do not occur in normal gauge theory!
2-Group of Gauge Transformations

Definition

Given $M$ with cell decomposition including $(V, E, F)$ as above, the **2-group of gauge transformations** is $\text{Gauge} = G^V$, which has:

- **objects** $\gamma : V \to G$
- **morphisms** $(\gamma, \chi)$ with $\chi : V \to H$
- **2-group structure** given by $\partial$ and $\triangleright$ acting pointwise as in $G$

Claim: there is a natural action of $\text{Gauge}$ on $\text{Conn}$:

$$\Phi : \text{Gauge} \to \text{End}(\text{Conn})$$
Definition (Gauge 2-Group Action - Part 1)

The action of Gauge on Conn is given by:

- An object $\gamma : V \to G$ of Gauge gives a functor $\Phi(\gamma) : \text{Conn} \to \text{Conn}$, “conjugation by $\gamma$” acting:
  - on objects $(g, h) \in \text{Conn}$ by:
    $$\Phi(\gamma)(g, h) = (\hat{g}, \gamma \triangleright h)$$
    where
    $$\hat{g}(e) = \gamma(s(e))^{-1}g(e)\gamma(t(e))$$
    and
    $$(\gamma \triangleright h)(e) = \gamma(s(e_1)) \triangleright h(f)$$
  - on morphisms $((g, h), \eta)$ by:
    $$\Phi(\gamma)((g, h), \eta) = ((\hat{g}, \gamma \triangleright h), \eta)$$
Definition (Gauge 2-Group Action - Part 2)

- A morphism \((\gamma, \chi)\) of \textbf{Gauge} gives a natural transformation

\[
\Phi(\gamma, \chi) : \Phi(\gamma) \Rightarrow \Phi(\gamma') : \text{Conn} \to \text{Conn}
\]

where \(\gamma' = \partial(\chi)\gamma\), defined as follows: for each object \((g, h) \in \text{Conn}\),

\[
\Phi(\gamma, \chi)(g, h) = ((\tilde{g}, \tilde{h}), \tilde{\eta})
\]

where

- \(\tilde{g}(e) = \gamma(s(e))^{-1}g(e)\gamma(t(e))\)
- \(\tilde{h}(f) = h(f)\)
- \(\tilde{\eta}(e) = \gamma(s(e))^{-1} \triangleright (\chi(s(e))^{-1}g \triangleright \chi(t(e)))\)

for each \(e \in E, f \in F\).

Goal: 2-Group analog of the theorem that \(A_0(M)\) is equivalent to a transformation groupoid, for this action.
Goal: We want to construct an analog $\mathbf{C} \sslash G$ for a transformation groupoid. It should be a 2-groupoid associated to a 2-group action.

$$\Phi : G \to \text{End}(\mathbf{C}) = \text{Cat}(\mathbf{C}, \mathbf{C})$$

It is easier if we understand $G$ as a group object in $\text{Cat}$, since the action can also be expressed (via “currying”):

$$\triangleright : G \times \mathbf{C} \to \mathbf{C}$$
The map $\rhd$ is an action if it satisfies:

$$\mathcal{G} \times \mathcal{G} \times \mathcal{C} \otimes \times \text{Id} \rightarrow \mathcal{G} \times \mathcal{C}$$

$$\text{Id} \times \rhd$$

$$\mathcal{G} \times \mathcal{C} \rhd$$

$$\rhd$$

$$\mathcal{G} \times \mathcal{C} \rightarrow \mathcal{C}$$

Idea: consider the construction of $S//G$ for a group action in diagrammatic terms in $\textbf{Set}$, and follow the same construction in $\textbf{Cat}$. 
Transformation Double Category

For a group action of $G$ on $S$, the transformation groupoid is constructed as a pullback in $\textbf{Set}$:

$$
\begin{array}{c}
S \parallel G \\ ^t \downarrow \\
G \times S \\
\downarrow \\
S
\end{array}
$$

This gives a set of morphisms, whose composition comes from the pullback square (using $s = \pi_2$):

$$
\begin{array}{c}
P \\
^\pi_1 \\
\downarrow
\end{array}
\begin{array}{c}
S \parallel G \\
^t \downarrow \\
S \\
_\pi_2 \\
\downarrow \\
s
\end{array}
$$
Transformation Double Category

The transformation double category associated to $\triangleright: \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ is constructed by the analogous pullbacks in $\text{Cat}$:
Description of the Transformation Double Category

Concretely, $\mathbf{C} \sslash \mathcal{G}$ is part of a category internal in $\mathbf{Cat}$. The category of objects is $\mathbf{C}$, with objects and morphisms:

$$
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow & & \downarrow \\
  (\gamma, x) & \xrightarrow{((\gamma, \eta), f)} & ((\partial \eta) \gamma, y) \\
  \gamma \triangleright x & \xrightarrow{(\partial \eta) \gamma} & \triangleright y
\end{array}
$$

The category of morphisms is $\mathbf{C} \sslash \mathcal{G}$. Its objects and morphisms are the vertical arrows and squares of:
The squares are diagonals of the naturality cubes:

(Note: the cube’s four side faces are themselves special cases of squares when $f = \text{Id}_x$ or $\eta = 1_H$.)
Folding to a 2-Category

Proposition

Claim: For a manifold $M$ (with cell decomposition $(V, E, F)$), the transformation double category $\text{Conn} \sslash \text{Gauge}$ is equivalent to the functor 2-category $\text{Hom}(\Pi_2(M, (V, E, F)), \mathcal{G})$.

To parse this, we must “fold” the double category $\mathcal{C} \sslash \mathcal{G}$ to give a 2-category $\widehat{\mathcal{C}} \sslash \mathcal{G}$ with:

- The same objects as $\mathcal{C} \sslash \mathcal{G}$ (hence of $\mathcal{C}$)
- Morphisms: Composites of horizontal and vertical morphisms of $\mathcal{C} \sslash \mathcal{G}$, i.e.:
  - morphisms of the object category
  - objects of the morphism category
- 2-Morphisms: squares of $\mathcal{C} \sslash \mathcal{G}$
Example 1: Connections on the Circle

Our general claim is that $\mathcal{A}_0(M, (V, E, F)) \cong \text{Conn} \sslash \text{Gauge}$. For the circle, we have worked this out in detail already.

$\mathcal{A}_0(S^1) = \text{Hom}(\Pi_2(S^1), \mathcal{G})$, with:

- **Objects**: Functors $F : \Pi_2(S^1) \to \mathcal{G}$, which are determined by $F(1) \in G$
- **Morphisms**: Natural transformations $n : F \Rightarrow F'$ determined by $\gamma \in G$ and $\eta \in H$
- **2-Morphisms**: Modifications $\phi : n \Rightarrow n'$ determined by $\chi \in H$

**Theorem**

*There is an equivalence of 2-groupoids $\mathcal{A}_0(S^1) \cong \hat{G} \sslash \mathcal{G}$.***
Example 1: Connection on a Circle
Example 1: Gauge Transformation on a Circle
Example 1: Gauge Modification on a Circle
Example 2: Connection on a Torus
Example 2: Gauge Transformation on a Torus
Example 2: Gauge Modification on a Torus