Extended TQFT in a Bimodule 2-Category

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Summary: Describe an Extended Topological Field Theory with topological action in terms of a factorization into classical field theory and quantization functor.

Definition

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \rightarrow \mathbf{2Hilb}$$

where $\mathbf{nCob_2}$ has

- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)
- **2-Morphisms**: n-dimensional cobordisms with corners

The related program of Freed, Hopkins, Lurie, Teleman aims to describe *local structure* of n-dimensional TQFT as a fully-extended ETQFT. That is, an n-functor from $nCob_n$ to **nAlg**.
Their program has two parts:

- A **classical field theory**, valued in *groupoids*
- A **quantization functor**, valued in n-algebras (roughly, monoidal n-vector spaces)

Goal: Define such an ETQFT by a factorization

$$Z_G = \Lambda \circ \mathcal{A}_0(-)$$

where

$$\mathcal{A}_0(-) : \mathbf{nCob_2} \rightarrow Span_2(\mathbf{Gpd})$$

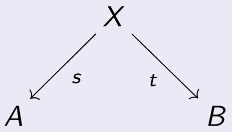
and

$$\Lambda : \mathrm{Span}_2(\mathbf{Gpd}) \rightarrow \mathbf{2Hilb}$$

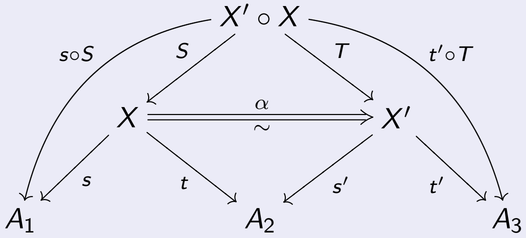
Definition (Part 1)

The bicategory $Span_2(\mathbf{Gpd})$ has:

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids:

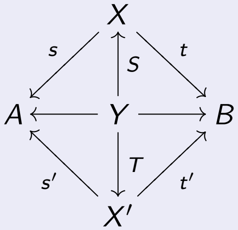


- Composition defined by *weak* pullback:



Definition (Part 2)

The **2-morphisms** of $Span_2(\mathbf{Gpd})$ are spans of *span maps*, commuting up to 2-cells of **Gpd**:



Composition is by weak pullback taken up to isomorphism.

Theorem

There is a monoidal structure on $Span_2(\mathbf{Gpd})$ *induced by the product in* **Gpd**, *with monoidal unit* 1.

Spans in Physics

- Span(**C**) is the *universal* 2-category containing **C**, and for which every morphism has a (two-sided) adjoint. The fact that arrows have adjoints means that *Span*(**C**) is a †-monoidal category. This is useful to describe quantum physics. (See Abramsky and Coecke, Vicary). (Compare also "dualizable" conditions for TQFT.)
- Physically, X will represent an object of *histories* leading the system A to the system B. Maps s and t pick the starting and terminating *configurations* in A and B for a given history (in the sense internal to **C**). (Adjointness corresponds to *time reversal* of histories.)
- Histories should be given an *action* by a Lagrangian functional. We'll see later how to incorporate this into Span(**Gpd**).

The "classical field theory" is a (topological) *gauge theory*, for gauge group G. The values are in the moduli space of connections:

Definition

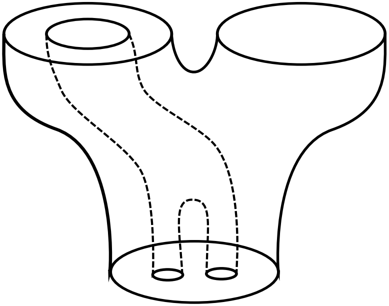
Given M, the groupoid $\mathcal{A}_0(M) = hom(\pi_1(M))/\!\!/G$ has:

- **Objects**: Flat connections on M
- **Morphisms** Gauge transformations

This induces a 2-functor:

$$\mathcal{A}_0(-) : \mathbf{nCob_2} \to Span_2(\mathbf{Gpd})$$

This fact uses that $\mathbf{nCob_2} \subset Span^2(\mathbf{ManCorn})$, consisting of double cospans:

$\mathbf{nCob_2}$	$Span^2(ManCorn)$
	$\begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A \coprod D) & \xleftarrow{i'_A \otimes i_D} & S^1 \coprod S^1 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle i_2} \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle i_1} \\ S^1 \coprod S^1 & \xrightarrow[i_2]{} & Y & \xleftarrow[i_1]{} & S^1 \end{array}$

(These form a "double bicategory", but it gives a bicategory since horizontal and vertical morphisms are composable.)

Theorem

There is a 2-functor ("2-linearization"):

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Hilb}$$

Where, recall:

Definition

2Hilb is the 2-category of 2-Hilbert spaces, which consists of:

- Objects: **Hilb**-enriched abelian $\star$-categories
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence abelian) functor.
- 2-Morphisms: Natural transformations

"Finite dimensional" (i.e. finitely generated) 2-Hilbert spaces are characterized by:

- Any fin. dim. 2-Hilbert space is equivalent to $\mathbf{Hilb}^k$ for some k (category of k-tuples of Hilbert spaces)
- 2-linear maps represented by a matrix of Hilbert spaces (acting by matrix multiplication with $\otimes$ and $\oplus$)
- natural transformations represented by a matrix of linear maps

If we have a little more structure, we have:

- Any monoidal 2-Hilbert spaces is equivalent to $Rep(\mathbf{G})$, the category of (continuous) unitary representations of some compact (super)groupoid

Conjecture (Baez, Baratin, Freidel, Wise):

- Any 2-Hilbert spaces is equivalent to $Rep(\mathcal{A})$ for some von Neumann algebra $\mathcal{A}$.

If $\mathbf{X}$ and $\mathbf{B}$ are (nice) groupoids, $f : \mathbf{X} \to \mathbf{B}$ gives restriction map $f^* = F \circ f : Rep(\mathbf{B}) \to Rep(\mathbf{X})$ and the *induced representation* of F along f:

$$f_* : Rep(\mathbf{X}) \to Rep(\mathbf{B})$$

is the two-sided adjoint of f^*.
In fact, the LEFT adjoint map f_* acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x) \cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

There is also a RIGHT adjoint:

$$f_!(F)(b) \cong \bigoplus_{[x] | f(x) \cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

There is the canonical *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \to f_*(F)(b)$$

given by the *exterior trace map* (which uses a modified group average in each factor):

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

Under this identification we get that f^* and f_* are ambidextrous adjoints.

Call the adjunctions in which f_* is left or right adjoint to f^* the *left and right adjunctions* respectively. We want to use the counit for the right adjunction, the evaluation map:

$$\eta_R(G)(x) : v \mapsto \bigoplus_{y|f(y)\cong x} (g \mapsto g(v))$$

and the unit for the left adjunction, which is determined by the action:

$$\epsilon_L(G)(x) : \bigoplus_{[y]|f(y)\cong x} g_y \otimes v \mapsto \sum_{[y]|f(y)\cong x} f(g_y)v$$

These define maps between $F(x)$ and $f_* f^* F(x)$.

(Note: there are canonical inner products around which make these maps *linear* adjoints.)

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Hilb}]$ (Unitary reps)
- Morphisms $\Lambda(\mathbf{A} \overset{s}{\leftarrow} \mathbf{X} \overset{t}{\rightarrow} \mathbf{B}) = t_* \circ s^* : \Lambda(\mathbf{A}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, S, T) = \epsilon_{L,T} \circ N \circ \eta_{R,S} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

Note: This Λ is a generalization of "degroupoidification" in the sense of Baez/Dolan. Both 1-morphisms and 2-morphisms use some form of "pull-push" process.

We'll consider extending the construction for Λ by replacing **2Hilb** with a bicategory of bimodules:

Definition

The bicategory $C^* - Bim$ has:

- **Objects**: C^*-algebras
- **Morphisms**: $Hom(A, B)$ consists of all (A, B)-Hilbert bimodules: Hilbert spaces with compatible (unitary) left action of A and right action of B
- **2-Morphisms**: Bimodule maps

via:

- von Neumann algebra: $\mathbf{B} \mapsto Rep(\mathbf{B})$
- 2-linear maps represented by *Hilbert bimodules*
- Natural transformations represented by bimodule maps

A span like:

$$
\begin{array}{ccc}
 & \mathcal{A}_0(\Sigma) & \\
 {\scriptstyle s} \swarrow & & \searrow {\scriptstyle t} \\
\mathcal{A}_0(S_1) & & \mathcal{A}_0(S_2)
\end{array}
$$

yields the span:

$$
\begin{array}{ccc}
 & Rep(C^*(\mathcal{A}_0(\Sigma))) & \\
 {\scriptstyle s^*} \nearrow & & \underset{t^*}{\overset{t_*}{\rightleftarrows}} \\
Rep(C^*(\mathcal{A}_0(S_1))) & & Rep(C^*(\mathcal{A}_0(S_2)))
\end{array}
$$

and thus

$$\Lambda(\mathcal{A}_0(\Sigma), s, t) = (t_* \circ s^*) : Rep(C^*(\mathcal{A}_0(S_1))) \to Rep(C^*(\mathcal{A}_0(S_2)))$$

Frobenius reciprocity (adjointness of t_* and t^*) says that if ρ is an irrep of $C^*(\mathcal{A}_0(S_1))$, the multiplicity of an irrep ϕ of $C^*(\mathcal{A}_0(S_2))$ in $\Lambda(\Sigma)(\rho)$ is the dimension of:

$$M(s,t,\rho,\phi) = Hom_{Rep(C^*(\mathcal{A}_0(\Sigma)))}(s^*\rho, t^*\phi)$$

Thus the functor $\Lambda(\Sigma)$ is given by tensoring with the bimodule:

$$B(s,t) = \bigoplus_{(\rho,\phi)} \rho \otimes_{C^*(\mathcal{A}_0(S_1))} M(s,t,\rho,\phi) \otimes_{C^*(\mathcal{A}_0(S_2))} \phi$$

But irreps of groupoids are classified by pairs $([g],\psi)$, where

- $[g]$ is an isomorphism class of object
- ψ is an irrep of $Aut(g)$

So if we specify $\rho = ([a_1],\rho)$, and $\phi = ([a_2],\phi)$ then $M(s,t,\rho,\phi)$ is:

$$M(s,t,([a_1],\rho),([a_2],\phi)) = \mathrm{hom}_{Rep(Aut(a_2))}(t_* \circ s^*(\rho),\phi)$$
$$\simeq \int^{\oplus}_{[x]\in \underline{(s,t)^{-1}([a_1],[a_2])}} \mathrm{hom}_{Rep(\mathrm{Aut}(x))}(s^*(\rho), t^*(\phi))$$

Given a 2-morphism in $\mathbf{nCob_2}$, we get a span of groupoid span maps:

$$\begin{array}{ccccc} & & \mathcal{A}_0(X) & & \\ & \overset{s}{\swarrow} & \uparrow S & \overset{t}{\searrow} & \\ \mathcal{A}_0(S_1) & \longleftarrow & \mathcal{A}_0(M) & \longrightarrow & \mathcal{A}_0(S_2) \\ & \underset{s'}{\nwarrow} & \downarrow T & \underset{t'}{\nearrow} & \\ & & \mathcal{A}_0(X') & & \end{array}$$

Then Λ gives rise to a bimodule map $B(s,t) \to B(s',t')$, given by the maps

$$M(s,t,\rho,\phi) \overset{(\epsilon_{L,T})_{\rho,\phi}}{\longrightarrow} M(s \circ S, t \circ T, \rho, \phi) \overset{N \circ (\eta_{R,S})_{\rho,\phi}}{\longrightarrow} M(s', t', \rho, \phi)$$

Theorem

The above construction gives an ETQFT valued in the bimodule category:

$$\hat{Z}_G : \mathbf{nCob_2} \rightarrow C^* - Bim$$

Idea: This describes the physics of a QFT on spacetimes with boundary:

- Algebras associated to boundaries describe symmetries
- Irreps (e.g. ρ and ϕ) are *superselection sectors*
- (A, B)-bimodules like $B(s, t)$ are Hilbert spaces for space with boundaries
- Bimodule maps describe (time)-evolution operators

We want a context to look at the twisted DW theory. The twisted theory has a "topological action" which depends on a class

$$[\omega] \in H^n_{grp}(G, U(1))$$

which we think of as represented by a particular cocycle

$$\omega \in Z^n(BG, U(1))$$

(Since n now matters, we'll stick to $n = 3$.)

Then the twisted form of the ETQFT will factor as:

$$Z^\omega_G = \Lambda^{U(1)} \circ \mathcal{A}^\omega_0(-)$$

But this factors through a different category. Which one?
The key idea is *transgression* of the cocycle on BG.

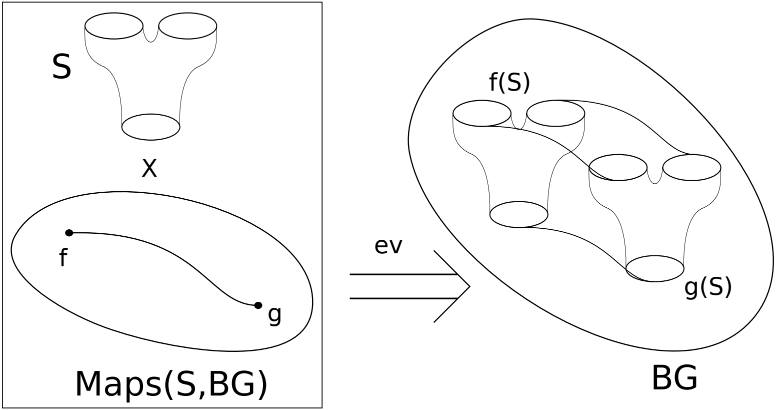
Recall that BG for a group(oid) G can be constructed as a simplicial complex with:

- A vertex (0-simplex) for each object of G
- An edge (1-simplex) for each morphism of G (group element)
- Higher cells for all composition relations, and so that BG has no higher homotopy groups

It is constructed so that $\Pi_1(BG) = G$, and we have that:

$$Hom(\Pi_1(M), G) \cong Maps_0(M, BG)$$

That is, flat connections on M correspond to homotopy classes of maps from M to BG.
Transgression of $\omega \in Z^3(BG, U(1))$ is a way to pull back the cocycle ω to the groupoids of connections.



There is the evaluation map:

$$ev : M \times Maps(M, BG) \rightarrow BG$$

If M is k-dimensional, $Im(M \times f) = f(M)$ is a (possibly degenerate) k-chain in BG. So we have a $(3-k)$-cocycle on $Maps(M, BG)$, the "transgression" of ω:

$$\tau_M(\omega) \in H^{3-k}(Maps(M, BG), U(1))$$

It is given by integrating ω:

$$\tau_M(\omega) = \int_M ev^*(\omega)$$

But since $Maps(M, BG)$ classifies the groupoid $\mathcal{A}_0(M)$, this is a $(3-k)$-cocycle in the groupoid cohomology!
This tells us the 2-category we need.

Definition (Part 1)

The monoidal 2-category $\mathrm{Span}(\mathbf{Gpd})^{U(1)}$ has:

- **Objects**: groupoids A equipped with 2-cocycle $\theta \in Z^2(A, U(1))$
- **1-Morphisms**: a morphism from (A, θ_A) to (B, θ_B) is a span of groupoids $A \stackrel{s}{\leftarrow} X \stackrel{t}{\rightarrow} B$, equipped with 1-cocycle $\alpha \in Z^1(X, U(1))$
- **2-morphisms**: a 2-morphism from (X, α, s, t) to (X', α', s', t') in $Hom((A, \theta_A), (B, \theta_B))$ is a class of spans of span maps $X \leftarrow Y \rightarrow X'$ equipped with 0-cocycle $\beta \in Z^0(Y, U(1))$, with equivalence taken up to β-preserving isomorphism of Y

But this is subject to some conditions...

Definition (Part 2)

- In any 1-morphism

$$(X, \alpha, s, t) : (A, \theta_A) \to (B, \theta_B)$$

  the cocycles satisfy

$$(s^*\theta_A) = (t^*\theta_B)$$

- In any 2-morphism

$$(Y, \beta, S, T) : (X_1, \alpha_1, s_1, t_1) \Rightarrow (X_2, \alpha_2, s_2, t_2)$$

  the cocycles satisfy

$$(S^*\alpha_1)(T^*\alpha_2)^{-1} = 1$$

In particular, $[s^*\theta_A] = [t^*\theta_B]$ and $[S^*\alpha_1] = [T^*\alpha_2]$.

Composition of:

$$(X_1, \alpha_1, s_1, t_1) : (A, \theta_A) \to (B, \theta_B)$$

and

$$(X_2, \alpha_2, s_2, t_2) : (B, \theta_B) \to (C, \theta_C)$$

at (B, θ_B) gives the same span of groupoids as in Span(**Gpd**).
The pullback groupoid's objects are triples (x_1, f, x_2) where
$f : t_1(x_1) \to s_2(x_2) \in B$. Its morphisms are:

$$\begin{array}{ccc} s_1(x_1) & \xrightarrow{f} & t_2(x_2) \\ {\scriptstyle s_1(g_1)}\downarrow & & \downarrow{\scriptstyle t_2(g_2)} \\ s_1(x_1') & \xrightarrow[f']{} & t_2(x_2') \end{array}$$

This groupoid gets the 1-cocycle

$$\alpha_1 \cdot \alpha_2 \cdot \theta_B$$

which assigns, to the morphism above, the value

$$\alpha_1(g_1) \cdot \alpha_2(g_2) \cdot \theta_B(f, f')$$

(Similar story for 2-morphism compositions)

Theorem

$\text{Span}(\textbf{Gpd})^{U(1)}$ *is a symmetric monoidal 2-category, and contains* $\text{Span}(\textbf{Gpd})$ *as a sub-(symmetric monoidal 2-category) consisting of those objects and morphisms with constant cocycles* $\theta = 1$, $\alpha = 1$, $\beta = 1$.

Idea: The 0-cocycles on 2-morphisms are the *Lagrangian*, or *action functional* on objects: connections on top-dimensional cobordism.
The twisted theory will factor as:

- $\mathcal{A}_0^{\omega}(-) : \textbf{3Cob}_\textbf{2} \rightarrow \text{Span}(\textbf{Gpd})^{U(1)}$
- $\Lambda^{U(1)} : \text{Span}(\textbf{Gpd})^{U(1)} \rightarrow \textbf{2Hilb}$

Note: only the "classical" part of this factorization depends on the choice of cocycle ω.

The Classical Field Theory

We can define the classical field theory, valued in groupoids carrying cocycles.

Definition

The for a fixed (compact) group G and group 3-cocycle ω, the classical field theory is a symmetric monoidal 2-functor:

$$\mathcal{A}_0(-)^{\omega} : \mathbf{3Cob_2} \to \mathrm{Span}(\mathbf{Gpd})^{U(1)} \tag{1}$$

which acts as follows:

- Objects: $\mathcal{A}_0(S)^{\omega} = (\mathcal{A}_0(S), \tau_S(\omega))$
- Morphisms: $\mathcal{A}_0(\Sigma : S_1 \to S_2)^{\omega} = (\mathcal{A}_0(\Sigma), \tau_{\Sigma}(\omega), i_1^*, i_2^*)$ (where the i_j are the inclusion maps of the S_j into Σ).
- 2-Morphisms: $\mathcal{A}_0(M : \Sigma \to \Sigma')^{\omega} = (\mathcal{A}_0(M), \tau_M(\omega), i^*, (i')^*)$, where again i and i' are inclusion maps of Σ and Σ' into M.

(To prove it is a well-defined 2-functor, the key is Stokes' theorem to get the compatibility conditions for the cocycles).

The Twisted Quantization Functor

Definition (Part 1)

Define the 2-functor

$$\hat{\Lambda}^{U(1)} : \mathrm{Span}(\mathbf{Gpd})^{U(1)} \to C^* - Bim$$

acts on objects by

$$\Lambda^{U(1)}(A, \theta_A) = \mathbb{C}^{\theta_A}(A)$$

Where $\mathbb{C}^{\theta_A}(A)$ is the algebra of functions on (morphisms of) the groupoid A with the "twisted multiplication":

$$(F \star_A G)(f) = \int_{g \in G} F(g) G(g^{-1} f) \theta_A(g, g^{-1} f)$$

(The usual groupoid algebra occurse when $\theta_A \cong 1$.)

Definition (Part 2)

To a morphism $(X, \alpha_X, s, t) : (A, \theta_A) \rightarrow (B, \theta_B)$ $\hat{\Lambda}^{U(1)}$ defines a bimodule representing the 2-linear map:

$$\hat{\Lambda}^{U(1)}(X, \alpha_X, s, t) = t_* \circ (M_{\alpha_X})^* \circ s^*$$

where $M_{\alpha_X} : \mathbb{C}^{s^*\theta_A}(X) \rightarrow \mathbb{C}^{t^*\theta_B}(X)$ is the isomorphism of these groupoid algebras induced by multiplication by α_X.

The point is that $M_{\alpha_X}(F)(g) = \alpha_X(g)F(g)$ is an algebra automorphism for $\mathbb{C}^{s^*\theta_A}(X)$. Note that this is the same as $\mathbb{C}^{t^*\theta_B}(X)$ since $s^*\theta_A = t^*\theta_B$. The main effect of this on the bimodule is to twist the *inner product* on the intertwiner spaces.

Definition (Part 3)

To a 2-morphism $(Y, \beta_Y, \sigma, \tau) : (X_1, \alpha_1, s_1, t_1) \Rightarrow (X_2, \alpha_2, s_2, t_2)$ assign the bimodule map corresponding to the natural transformation:

$$\hat{\Lambda}^{U(1)}(Y, \beta_Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N_{\beta_Y} \circ \eta_{R,\sigma}$$

from $(t_1)_* \circ (M_{\alpha_1})^* \circ s_1^*$ to $(t_2)_* \circ (M_{\alpha_2})^* \circ s_2^*$ using the "twisted form" of the Nakayama isomorphism:

$$N_{\beta_Y} : \sigma_* \circ (M_{\sigma^*\alpha_1})^* \circ \sigma^* \Longrightarrow \tau_* \circ (M_{\tau^*\alpha_2})^* \circ \tau^*$$

relating the (α-twisted) forms of the left and right adjunction, at $y \in Y$ by:

$$N_{\beta_Y} : \bigoplus_{[y]|f(y)\cong x} \phi_y \mapsto \bigoplus_{[y]|f(y)\cong x} \frac{\beta_Y(y)}{\#Aut(y)} \sum_{g \in Aut(x)} g \otimes \phi_y(g^{-1})$$

Theorem

Given a finite gauge group G and 3-cocycle $\omega \in Z^3(BG, U(1))$, the symmetric monoidal 2-functor

$$Z_G^{\omega} = \Lambda^{U(1)} \circ \mathcal{A}_0(-)^{\omega} : \mathbf{3Cob_2} \rightarrow \mathbf{2Hilb}$$

reproduces the Dijkgraaf-Witten (DW) model with twisting cocycle ω.

When G is any compact Lie group, we will get an analogous $C^* - Bim$-valued ETQFT.

$$\hat{Z}_G^{\omega} : \mathbf{3Cob_2} \rightarrow C^* - Bim$$

Aim: Because the DW model is the discrete form of Chern-Simons theory, this should describe the local structure of CS theory with topological action. The twisting of N_{β_Y} gives the action in the path integral.

Extending to cover (compact) Lie groups, some formulas change, replacing $\oplus$ with $\int^{\oplus}$, etc. But:
If $G = SU(2)$, $\mathcal{A}_0(S^1) = SU(2)/\!\!/SU(2)$. The irreducible objects of $Rep(\mathcal{A}_0(S^1))$ (or reps of the groupoid algebra) are given by:

- conjugacy class $[g]$ of $SU(2)$
- representation of stabilizer of $[g]$: $U(1)$ ($SU(2)$ if $[g] = \pm e$):

Take the circle as boundary around an excised point particle: a conjugacy class in SU(2) is an angle in $[0, 2\pi]$, which is the *mass* m of particle; an irrep of $U(1)$ is labelled by an integer, the *spin* of a particle.

Generalization: Span(**Gpd**) is naturally a 2-category, so our construction can only give an ETQFT down to codimension 2.
To give better invariants for 4-manifolds, we perhaps should use a theory whose moduli space is valued in **2Gpd**... Higher gauge theory.
For a 2-group $\mathcal{G}$, define a 3-functor

$$Z_{\mathcal{G}} : \mathbf{nCob_3} \to \mathbf{3Vect}$$

factoring through a classical moduli space:

$$\mathcal{A}_0^{(2)} = 2Fun[\Pi_2(-), \mathcal{G}]$$

The 2-functor 2-groupoid, understood as flat 2-connections, gauge transformations, and "gauge modifications".