Extended TQFT From Gauge THeory

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Definition

A Topological Quantum Field Theory is a monoidal functor:

$$Z : \mathbf{nCob} \to \mathbf{Vect}$$

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Vect}$$

where $\mathbf{nCob_2}$ has

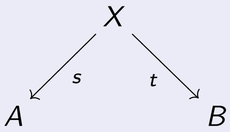
- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)
- **2-Morphisms**: n-dimensional cobordisms with corners

We'll construct an ETQFT by factoring through a 2-category $Span(\mathbf{Gpd})$, then applying some universal process.

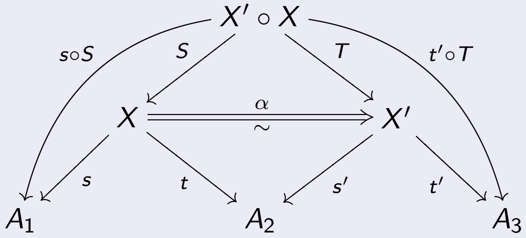
Definition (Part 1)

The bicategory $Span_2(\mathbf{Gpd})$ has:

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids:

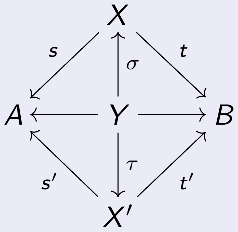


- Composition defined by *weak* pullback:



Definition (Part 2)

The **2-morphisms** of $Span_2$(**Gpd**) are spans of *span maps*, commuting up to 2-cells of **Gpd**:



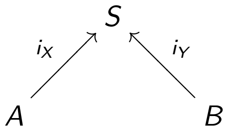
Composition is by weak pullback taken up to isomorphism.

Theorem

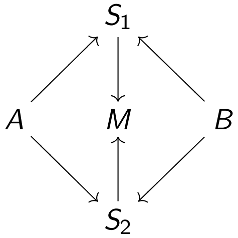
There is a monoidal structure on $Span_2$(**Gpd**) *induced by the product in* **Gpd**, *with monoidal unit* 1.

(Note: Roughly, $Span_2$(**C**) will be the universal 2-category containing **C** in which morphisms have ambidextrous adjoints.)

Cobordisms can be seen as *cospans* of manifolds, with inclusions:



A cobordism between two cobordisms is a cospan of cospan maps:



(Note there are complications due to the fact that $\mathbf{nCob_2}$ is a *cubical* weak 2-category.)

For finite gauge group G, we get a functor:

$$\mathcal{A}_G : \mathbf{nCob_2} \to Span(\mathbf{Gpd})$$

Definition

Moduli space for *gauge theory*, for (finite) gauge group G. Given M, the groupoid $\mathcal{A}_G(M) = Fun(\pi_1(M), G)$ has:

- **Objects**: Flat connections on M (functors)
- **Morphisms** Gauge transformations (natural transformations)

("Secretly" the groupoid is representing a *stack*. This is a standard situation for moduli spaces supporting symmetries.)

A connection on the cobordism $S : X \rightarrow Y$ in $\mathbf{nCob_2}$ can be pulled back along boundary inclusions by $(i_X)^\star$ and $(i_Y)^\star$, hence there is a span of the groupoids of flat connections:

$$\mathcal{A}_G(X) \xleftarrow{(i_X)^*} \mathcal{A}_G(S) \xrightarrow{(i_Y)^*} \mathcal{A}_G(Y)$$

Theorem

$\mathcal{A}_G(-)$ defines a contravariant functor $\mathbf{ManCorn} \rightarrow \mathbf{Gpd}$), and a covariant functor $\mathbf{nCob_2} \rightarrow Span(\mathbf{Gpd})$.

Think of $\mathcal{A}_G(S)$ as a space (*stack*) of *histories*; intuitively s and t pick the starting and terminating *configuration* in A and B - compatible with gauge symmetry.

Goal: Using the induced 2-functor $\mathcal{A}_G(-) : \mathbf{nCob_2} \rightarrow Span_2(\mathbf{Gpd})$, get an ETQFT $Z_G : \mathbf{nCob_2} \rightarrow \mathbf{2Vect}$.

Theorem

There is a 2-functor ("2-linearization"):

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

Where, recall:

Definition

2Vect is the 2-category of 2-vector spaces, which consists of:

- Objects: $\mathbb{C}$-linear abelian category, generated by simple objects
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence abelian) functor.
- 2-Morphisms: Natural transformations

Finite dimensional 2-vector spaces all look like $\mathbf{Vect}^k$, and 2-linear maps have a matrix representation. (Analogous examples occur for infinite dimensional 2-vector spaces).

Lemma

If $\mathbf{B}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$ *is a KV 2-vector space.*

The generators of $[\mathbf{B}, \mathbf{Vect}]$ are irreducible reps - labeled by $([b], V)$, where $[b] \in \underline{\mathbf{B}}$ and V an irreducible rep of $Aut(b)$.

Theorem

If $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \to \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \to \Lambda(\mathbf{X})$$

with $f^* F = F \circ f$ *and (the restricted representation along* f*)*

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{B})$$

the induced representation of F *along* f*. Furthermore,* f_* *is the two-sided adjoint to* f^**.*

In fact, the LEFT adjoint map f_* acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x) \cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a (left) *Kan extension* of the functor F along f.

There is also a RIGHT adjoint (right Kan extension):

$$f_!(F)(b) \cong \bigoplus_{[x] | f(x) \cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

There is the canonical *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \rightarrow f_*(F)(b)$$

given by the *exterior trace map* (which uses a modified group average in each factor):

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

Under this identification we get that f^* and f_* are ambidextrous adjoints.

Call the adjunctions in which f_* is left or right adjoint to f^* the *left and right adjunctions* respectively. We want to use the counit for the right adjunction, the evaluation map:

$$\eta_R(G)(x) : v \mapsto \bigoplus_{y|f(y)\cong x} (g \mapsto g(v))$$

and the unit for the left adjunction, which is determined by the action:

$$\epsilon_L(G)(x) : \bigoplus_{[y]|f(y)\cong x} g_y \otimes v \mapsto \sum_{[y]|f(y)\cong x} f(g_y)v$$

These define maps between $F(x)$ and $f_* f^* F(x)$.

(Note: there are canonical inner products around which make these maps *linear* adjoints.)

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathbf{a}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

$\Lambda(X, s, t)$ can be represented by the matrix with coefficients:

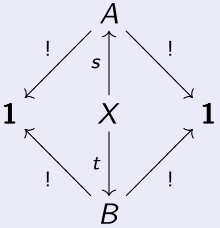
$$\Lambda(X, s, t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$
$$\simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

This is an *intertwiner space* for the groupoid representations. The 2-morphisms give (component-wise) linear maps between intertwiner spaces.

In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

Theorem

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*



where $\mathbf{1}$ *is the (terminal) groupoid with one object and one morphism,* Λ *on 2-morphisms is just the degroupoidification functor* D *of Baez and Dolan.*

Theorem

For any finite group G, the 2-functor

$$Z_G = \Lambda \circ \mathcal{A}_G$$

is an extended TQFT.

That is, a cobordism becomes:

$$[\mathcal{A}_G(X), \mathbf{Vect}] \stackrel{\Lambda(\mathcal{A}_G(S),(i_X)^*,(i_Y)^*)}{\longrightarrow} [\mathcal{A}_G(Y), \mathbf{Vect}]$$

and similarly for 2-morphisms.

Corollary

$Z_G = \Lambda \circ \mathcal{A}_G$ gives the Dijkgraaf-Witten model when $n = 3$, for closed manifolds.

Lie Groups and Measurable Groupoids

Z_G for G a compact Lie group requires *measured groupoids*

Duplicating the above requires some changes:

- Ambi-adjunction requires **Hilb** instead of **Vect** in infinite-dim setting
- Direct sums become direct integrals - which are not (co)limits
- Push-forward is not just Kan extension of functors

Any 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ is naturally isomorphic to a map acting by an $m \times n$ matrix:

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,n} \\ \vdots & & \vdots \\ V_{m,1} & \dots & V_{m,n} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^n V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^n V_{m,i} \otimes W_i \end{pmatrix}$$

When the entries are finite-dimensional vector spaces, this explains why T has a two-sided adjoint.
T^* is the $n \times m$ matrix with $(T^*)_{i,j} = (T_{i,j})^*$, the dual of the corresponding entry it the transpose of T. The adjoint is 2-sided because $(V_{i,j})^{**} \cong V_{i,j}$ is canonical: the category **FinVect** is *reflexive*.

This isn't true for infinite-dimensional vector spaces, but it is for Hilbert spaces (**Hilb** is reflexive). So to stay closed under composition, in infinite-dimensions, 2-Vector spaces must be generalized to 2-Hilbert spaces.

Consider categories like:

Definition

If (X, μ) is a measurable space **Meas(X)** is the category with:

- Objects: *measurable fields of Hilbert spaces* on $(X, \mathcal{M})$: i.e. X-indexed families of Hilbert spaces $\mathcal{H}_x$ with a Hilbert space of *measurable sections* (satisfying certain properties)
- Morphisms: *measurable fields of bounded linear maps* between Hilbert spaces, $f_x : \mathcal{H}_x \to \mathcal{K}_x$ so that $||f||$ (the operator norm of f) is measurable.

This is the equivalent of a *measurable function*. Imposing that sections and norms be L^2 condition gives a categorification of $L^2(X, \mu)$.

Definition

There is a locale MX whose "open sets" are the measurable sets of X, and whose morphisms are inclusions *up to almost everywhere*. This becomes a Grothendieck site where an "open cover" is a usual cover, *up to almost everywhere*.

Then a *measurable sheaf of Hilbert spaces* is a sheaf of Hilbert spaces on on MX, and these form a category $MSh(X)$.

Theorem (Wendt)

The category of measurable sheaves $MSh(X)$ is equivalent to the internal category $Hilb[Sh(X)]$ of Hilbert spaces in the topos $Sh(X)$ of (set-valued) sheaves on MX.

Theorem

A measurable field of Hilbert spaces on (X, μ) determines a measurable sheaf by direct integration: given a measurable $U \subset X$, this assigns

$$\int_U^{\oplus} d\mu(x)\mathcal{H}_x$$

where the direct integral is a Hilbert space of sections with inner product

$$\langle \phi, \psi \rangle = \int_U d\mu(x) \langle \phi_x, \psi_x \rangle$$

This is the equivalent of the matrix of vector spaces for a 2-linear map. It is still a conjecture that all suitable functors are of this form.

Question: How do such functors arise?

Definition

A *disintegration* between two measure spaces consists of:

- A measurable function $f : (X, \mathcal{M}, \mu) \rightarrow (Y, \mathcal{N}, \nu)$
- A family $(X_y, \mathcal{M}_y, \mu_y)_{y \in Y}$ where:
  - $X_y = f^{-1}(y)$
  - $\mathcal{M}_y = \{A \cap X_y | A \in \mathcal{M}\}$
  - μ_y is a measure on X_y

satisfying some obvious properties.

Theorem (Wendt)

Given a disintegration $f : (X, \mu) \rightarrow (B, \nu)$*, there is an adjoint pair of functors*

$$MSh(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} MSh(Y)$$

We need (groupoid-)equivariant version of this theorem.

Definition

A measurable groupoid is a groupoid internal to **Msble**, the category of measurable spaces and measurable functions.

Definition

If $\mathcal{G} = (G_0, G_1)$ is a measurable groupoid, a **groupoid measure** on $\mathcal{G}$ consists of:

- A measure μ on the space of objects
- A (measurable, left) *Haar system*: for each $x \in G_0$, a measure ν_x on the space of morphisms into x, $t^{-1}(x)$ such that
  - the choice ν_x is measurable: for any measurable function $f : G_1 \to \mathbb{C}$, the function
    $$x \mapsto \int_{t^{-1}(x)} f(g) d\nu_x(g) \tag{1}$$
    is measurable
  - the ν_x are left-invariant: for any $g \in G_1$, and measurable $f : G_1 \to \mathbb{C}$
    $$\int f(gh) d\nu_{s(g)}(h) = \int f(h) d\nu_{t(g)}(h) \tag{2}$$

To define Λ for measure groupoids, we again want:

$$\Lambda(G) = Rep(G)$$

A representation ρ of a measure groupoid $s, t : G_1 \rightarrow G_0$ is defined on a measurable field of Hilbert spaces $\mathcal{H}$ on G_0. It gives a functor $R : \mathbf{G} \rightarrow \mathbf{Hilb}$ with $R(x) = \mathcal{H}_x$, the fibre at each $x \in M$, and an isomorphism $R(g)$ for each $g : x \rightarrow y$. (But not all functors are *measurable* representations).

Definition

$Rep(\mathbf{G})$, the **category of representations** of $\mathbf{G}$, has

- *Objects*: Measurable representations of $\mathbf{G}$
- *Morphisms*: Intertwiners: i.e. measurable natural transformations between functors $n : \rho \rightarrow \rho'$

(A natural transformation is measurable when it determines a measurable field of linear maps over G_0.)

Theorem

A representation of G on a measurable field $\mathcal{H}$ of Hilbert spaces determines an equivariant sheaf of Hilbert spaces by direct integration.

Then we hope to have the following:

Proposition

The category $Rep(G)$ is equivalent to the internal category $Hilb[EMSh(G)]$ of Hilbert spaces in the topos of equivariant measurable sheaves on MG_0.

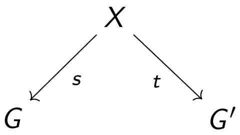
And

Proposition

Given a disintegrating functor $f : G \to G'$ between measure groupoids , there is a (bi-)adjoint pair of functors

$$EMSh(G) \underset{f_*}{\overset{f^*}{\rightleftarrows}} EMSh(G')$$

Given the above, we would define Λ as before, so that a span



has

$$\Lambda(X, s, t) = t_* \circ s^* : \Lambda(G) \longrightarrow \Lambda(G')$$

and for a 2-cell $Y : X \to X'$ given by

$$\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \to (t')_* \circ (s')^*$$

using the analog of the Nakayama isomorphism:

$$N : \int_{[x]|f(x)\cong b}^{\oplus} \phi_x \mapsto \int_{[x]|f(x)\cong b}^{\oplus} \frac{1}{vol(Aut(x))} \int_{g \in Aut(b)} g \otimes \phi_x(g^{-1})$$

Applying the above to ETQFT: follow the same prescription:

Example

Interesting case is $G = SU(2)$. The topology generates measurable sets to make $SU(2)$ a regular Borel space, with Haar measure μ.
The (measurable) groupoid

$$\mathcal{G} = Z_{SU(2)}(S^1) = SU(2) /\!\!/ SU(2)$$

gets a groupoid measure:

- **Measure**: $Ob(\mathcal{G}) = SU(2)$ gets the Haar measure μ
- **Haar System**: For $g \in Ob(\mathcal{G})$, we always have $t^{-1}(g) \cong SU(2)$, which also gets $\nu_g = \mu$

(Note: $vol(G /\!\!/ G) = 1$, as we've fixed normalization of μ)

We can get reps of $\mathcal{G}$ by integrating those indexed by $([g], V)$ for $g \in SU(2)$ and V an irrep of $Stab(g)$ ($SU(2)$ or $U(1)$).

Cobordisms of 2 or 3 dimensions are trickier:

For connected cobordisms, all groupoids in our construction are equivalent to ones of the form $\mathcal{A}_G(X)/\!\!/G$.
So we can always take $\nu_x = \mu$, Haar measure on G. But:

- There is a canonical measure on $\mathcal{A}_G(B)$ for 2-manifold S, the *Goldman measure*... but this is nontrivial!
- There is no *canonical* measure on $\mathcal{A}_G(M)$ for 3-manifold M!

To assign measures to $\mathcal{A}_G(X)$ in dimension 3 or higher, we must *change the cobordism category*.
Need cobordisms to be decorated with extra data, sufficient to determine a measure. (e.g. specified paths which determine a presentation of the fundamental groupoid)

The construction for Λ can also be described using:

- *Rep*(**B**) $\mapsto$ Category of reps of von Neumann algebra associated to **B** (including groupoid algebras)
- 2-linear maps represented by *Hilbert bimodules*, given by induction and restriction
- Natural transformations represented by bimodule maps

This relates to a conjecture (Baez, Baratin, Freidel, Wise) that *infinite-dimensional 2-Hilbert spaces* are representation categories for v.N.-algebras.
"Physically", this describes the quantum mechanics of systems with boundary.