# Groupoids of Connections and Higher-Algebraic QFT

Jeffrey C. Morton

Department of Mathematics
University of Western Ontario

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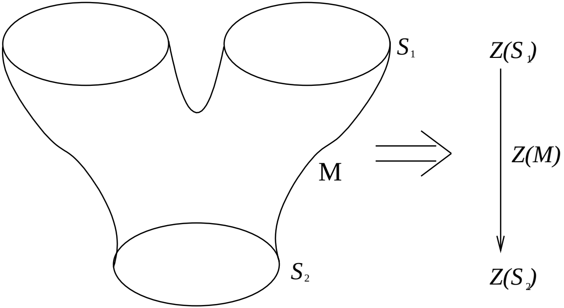
# Outline

1. TQFT and ETQFT
2. Groupoids of Connections
   - Groupoids and Moduli Spaces
   - Example: $S^1$
3. Constructing $Z_G$
   - 2-Vector Spaces for Manifolds
   - $Z_G$: 2-Linear Maps for Cobordisms
   - $Z_G$: 2-Morphisms
4. Physics - Sort Of

A Topological Quantum Field Theory can be seen as a monoidal functor:

$$Z_G : \mathbf{nCob} \to \mathbf{Vect}$$




In particular:

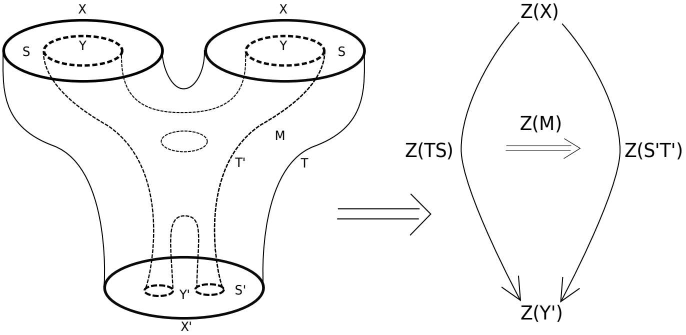
$$Z(M_2 \circ M_1) = Z(M_2) \circ Z(M_1)$$

and

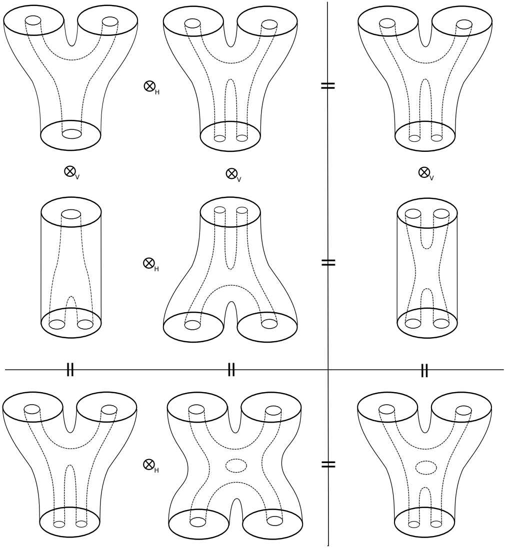
$$Z(S_1 \amalg S_2) = Z(S_1) \otimes Z(S_2) \text{ and } Z(\varnothing) = \mathbb{C}$$

We'll see that for each (finite, or compact Lie) group $G$, there is an *Extended TQFT*, namely a (monoidal) 2-functor:

$$Z_G : \mathbf{nCob_2} \longrightarrow \mathbf{2Vect}$$



# Cobordisms of cobordisms form a 2-category $\mathbf{nCob}_2$:



### Definition

A **2-Vector space** is a $\mathbb{C}$-linear abelian category generated by simple elements. A 2-linear map is an exact $\mathbb{C}$-linear functor.

Finite-dimensional 2-vector spaces are all equivalent to $\mathbf{Vect}^k$. 2-linear maps then look like:

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^{k} V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^{k} V_{l,i} \otimes W_i \end{pmatrix}$$

There are also *natural transformations* between 2-linear maps, which look like matrices with components $\alpha_{i,j} : V_{i,j} \rightarrow V'_{i,j}$.

A *groupoid* is a category in which all morphisms are invertible (a "many-object group", as a category is a "many-object monoid"). In a *Lie groupoid*, Ob and Mor $= \cup_{x,y} hom(x,y)$ are manifolds (and source, target, identity maps are surjective submersions).

If $X$ is a set, and a group $G$ acts on $X$, there is an *action groupoid* $X /\!\!/ G$ with:

- Objects: elements of $X$
- Morphisms: triples $(x, g, y)$ where $gx = y$ This groupoid, up to equivalence of groupoids, represents a *quotient stack*.

Two interesting moduli spaces:
• connections on a manifold $M$: $\mathcal{A}(M)$
• *flat* connections on $M$: $\mathcal{A}_0(M)$
Both are acted on by gauge transformations. We will mostly consider:

$$\mathcal{A}_0(M)/\!\!/\mathcal{G}$$

$\Pi_1(M)$ has objects $x \in M$ and morphisms homotopy classes of paths.
The groupoid of *flat* connections is equivalent to the functor category:

$$\mathcal{A}_0(B) = \mathrm{Fun}(\Pi_1(B), G)$$

(Gauge transformations are natural transformations between these functors).

For example, if $B = S^1$, $\Pi_1(S^1) \simeq \mathbb{Z}$. A $G$-connection $g$ is specified by the holonomy $g(1) \in G$. A natural transformation from $g$ to $g'$ is given by $h \in G$, such that $g' = hgh^{-1}$. So then:

$$\mathcal{A}_0(S^1) \simeq G /\!\!/ G$$

is equivalent to the groupoid with:

- Objects: conjugacy classes $[g]$ of $G$
- Morphisms: only isotopy subgroups $Aut(g)$ for each $[g]$

## Lemma

*If* $\mathbf{X}$ *is a groupoid, the functor category* $Rep(\mathbf{X}) = [\mathbf{X}, \mathbf{Vect}]$ *is a 2-vector space.*

Later on, *2-Hilbert space* structure will come from a "measure" on $\underline{\mathbf{X}}$, given using *groupoid cardinality*

$$|\mathbf{X}| = \sum_{[x]} \frac{1}{|Aut(x)|}$$

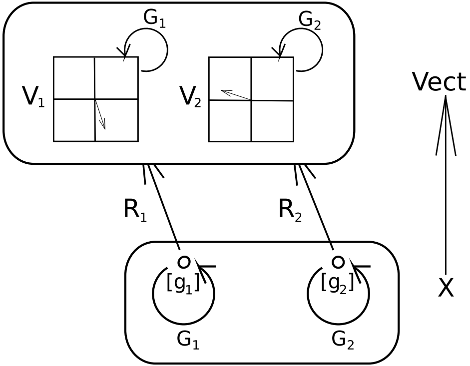
or the analog for differentiable stacks (Weinstein) from the "volume form":

$$vol(\mathbf{X}) = \int_{\underline{X}} \left( \int_{Aut([x])} d\nu \right)^{-1} d\mu$$

The methods used can also be used to apply to any theory whose *states* and *histories*, and their *symmetries* give moduli stacks of finite total volume. Here, these are connections and gauge transformations.
To build $Z_G : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$, use a topological gauge theory with gauge group $G$ (assume $G$ finite, or compact Lie). Flat $G$-connections on manifolds can be specified by holonomies along paths.
Then the 2-vector space $Z_G(B)$ is:

$$Z_G(B) = Rep(\mathcal{A}_0(B)) = [\mathcal{A}_0(B), \mathbf{Vect}]$$

Suppose $B = S^1$. We get $Z_G(S^1) = [\mathcal{A}(S^1), \mathbf{Vect}] \simeq [G /\!\!/ G, \mathbf{Vect}]$. This gives a vector space for each $[g] \in G$ and an isomorphism for each conjugacy relation:

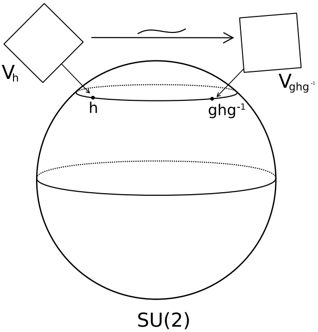



So that

$$Z_G(S^1) \simeq \prod_{[g]} \mathbf{Rep(Aut([g]))}$$

So any 2-vector in this 2-vector space is a direct sum of irreducible

A physically interesting case is $G = SU(2)$. The irreducible (basis) objects of $Z_{SU(2)}(S^1) \simeq [SU(2)/\!/SU(2), \mathbf{Vect}]$ amount to a choice of conjugacy class in $SU(2)$ (i.e. $\phi \in [0, 2\pi]$ and representation of stabilizer subgroup ($U(1)$ if $m \neq 0$, or $SU(2)$ if $m = 0$).




A general object corresponds to some coherent sheaf of vector spaces on $SU(2)/\!/SU(2)$ (i.e. equivariant).

A cobordism between manifolds can be expressed as a diagram:

$$B \xleftarrow{i} S \xrightarrow{i'} B'$$

which gives a diagram of the groupoids of connections:

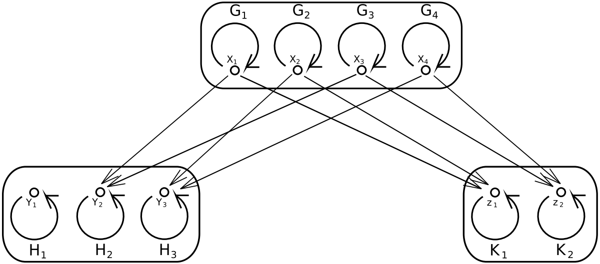
$$\mathcal{A}_0(B) \xleftarrow{i^\star} \mathcal{A}_0(S) \xrightarrow{(i')^\star} \mathcal{A}_0(B')$$

since both connections and gauge transformations on $S$ can be restricted along the inclusion maps $i$ and $i'$.

So we have:

$$Z_G(B) \xrightarrow{p^*} [\mathcal{A}_0(S), \mathbf{Vect}] \xleftarrow{(p')^*} Z_G(B')$$

where $p^*$ is the *pullback* 2-linear map, taking $F : \mathcal{A}_0(B) \to \mathbf{Vect}$ to $(F \circ p) : \mathcal{A}_0(S) \to \mathbf{Vect}$. Likewise $(p')^* : Z_G(B') \to [\mathcal{A}_0(S), \mathbf{Vect}]$. To push a 2-vector in $Z_G(B)$ to one in $Z_G(B')$ involves a (direct) sum over all "histories" - i.e. connections which restrict to $a$ and $a'$, as in this diagram:



Then picking basis elements $(a, W) \in Z_G(B)$ and $(a', W') \in Z_G(B')$, we get

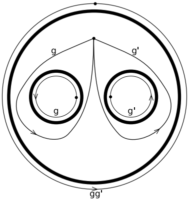
$$Z_G(S)_{(a,W),(a',W')} = \bigoplus_{[s]} \hom_{Rep(\mathrm{Aut}(s))}[p^*(W), (p')^*(W')]$$

for objects $s$ with $(p, p')(s) = (a, a')$.
(By Schur's lemma, this counts the multiplicity of the irrep $W'$ in $(p')_* \circ p^* W$.)
So the adjoint 2-linear map

$$(p')_* : \left[\mathcal{A}_0(S), \mathbf{Vect}\right] \to Z_G(B')$$

pushes forward a 2-vector $p^* F \in Rep(\mathcal{A}_0(S))$ to the *induced representation* in $Rep(\mathcal{A}_0(B'))$.

Suppose $Y : S^1 + S^1 \to S^1$ is the "pair of pants":

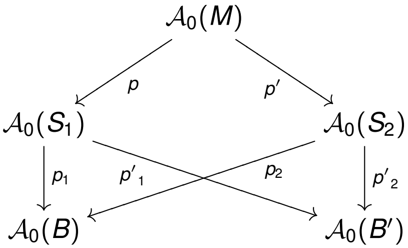



Then we have the diagram:

$$\begin{array}{ccc} & (G \times G)/\!\!/G & \\ \swarrow^{\Delta} & & \searrow^{m} \\ (G/\!\!/G)^2 & & G/\!\!/G \end{array} \tag{1}$$

$Z_G(Y)$ sends a representation over $([g],[g'])$ to one with nontrivial reps over $[gg']$ for any representatives $(g,g')$.

Given a cobordism with corners between two cobordisms with the same source and target: there is a tower of groupoids:




Then we get:

$$Z_G(M) : Z_G(S_1) \to Z_G(S_2)$$

a natural transformation whose components are *linear* maps:

$$Z_G(M)_{([a],W),([a'],W')} : \bigoplus_{[s_1]} \hom_{Rep(\mathrm{Aut}(s_1))}[p_1^*(W), p_2^*(W')]$$

$$\to \bigoplus_{[s_2]} \hom_{Rep(\mathrm{Aut}(s_2))}[{p'}_1^*(W), {p'}_2^*(W')]$$

The natural transformation

$$Z_G(M)_{([a],W),([a'],W')} : \bigoplus_{[s_1]} \hom_{Rep(\mathrm{Aut}(s_1))}[p_1^*(W), p_2^*(W')] \rightarrow \bigoplus_{[s_2]} \hom_{Rep(\mathrm{Aut}(s_2))}[p'^*_1(W), p'^*_2(W')]$$

has components which are given by:

$$Z_G(M)_{([a],W),([a'],W'),(s_1,s_2)}(f) = |\widehat{(s_1,s_2)}| \sum_{g \in \mathrm{Aut}(s_2)} gfg^{-1}$$

where $\widehat{(s_1,s_2)}$ is a subgroupoid of $\mathcal{A}_0(M)$, the “essential preimage” of $(s_1,s_2)$ under $(p,p')$, and $|\cdot|$ is the groupoid cardinality (or stack volume).
(This comes from an analogous “pull-push” operation: cf Baez and Dolan, “Groupoidification”.)

**Theorem**

*The construction we've just seen gives a 2-functor*

$$Z_G : \mathbf{nCob_2} \longrightarrow \mathbf{2Vect}$$

*(that is, an Extended TQFT).*

For physics, we really want **2-Hilbert spaces**: **Hilb**-enriched abelian $\star$-categories with all limits. Generated by simple objects (i.e. ones where $\hom(x,x) \cong \mathbb{C}$.

Typical example: a category of **fields of Hilbert spaces**, ($\mathcal{H}$ on a measure space $(X,\mu)$ consists of an $X$-indexed family of Hilbert spaces $\mathcal{H}_x$ (together with a good space of sections).

Morphisms are (certain) **fields of bounded operators** $\phi : \mathcal{H} \rightarrow \mathcal{K}$, with $\phi_x \in \mathcal{B}(\mathcal{H}_x, \mathcal{K}_x)$ preserving good sections.

**2-linear maps**: $\mathbb{C}$-linear additive $\star$-functors.

$\Phi_{\mathcal{K},\mu}$ : **Meas(X)** $\rightarrow$ **Meas(Y)** is specified by:

- a field of Hilbert spaces $\mathcal{K}_{(x,y)}$ on $X \times Y$
- item a $Y$-family $\{\mu_y\}$ of measures on $X$, where:

$$\Phi_{\mathcal{K},\mu}(\mathcal{H})_y = \int_X^{\oplus} \mathcal{H}_x \otimes \mathcal{K}_{(x,y)} \mathrm{d}\mu_y(x)$$

When $B = B' = \emptyset$, so that $\mathcal{A}_0(B) = \mathcal{A}_0(B') = 1$, the terminal groupoid, with $Rep(1) =$ **Vect**. Then the extended TQFT reduces to a TQFT. For $G$ is a finite group, this theory reproduces the (untwisted) Dijkgraaf-Witten model. If $G$ is compact Lie, this is *BF theory*.
For $B \neq \emptyset$, this describes a TQFT coupled to boundary conditions—"matter". Take the circle as boundary around an excised point particle!
If $G = SU(2)$ and $n = 3$, this depicts particles classified by mass ($m \in [0, 2\pi]$) and spin (unitary group representations) propagating on a background described by 3D quantum gravity (a BF theory in 3D). If $n = 4$, this is a limit of gravity as Newton's $G \to 0$.