2-Linearization In Physics and Topology

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Motivation: Categorify a quantum mechanical description of states and processes.
Applications Foundational physics such as quantum harmonic oscillator; Witten-type ETQFT (help interpret physical examples).
Categorification involves replacing set-based structures with category-based structures. That is, by replacing the category **Set** with the 2-category **Cat** (or **SmallCat**). There are two obvious approaches to how the original structure reappears (apart from "by analogy"):

1. "Quotient": from the object/morphism level (Grothendieck ring - e.g. categorified $\mathfrak{sl}_2$)
2. "Substructure": At the morphism/2-morphism level (the "microcosm principle")

There are more possibilities when going to *n*-categories.

The following is an example of the last type:

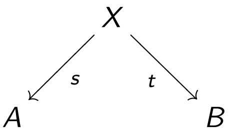
Theorem

There is a 2-functor ("2-linearization"):

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

This is a categorification (sense 2) of the "degroupoidification" functor $D : Span_1(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$ of Baez and Dolan (which itself gives an example of sense 1!)

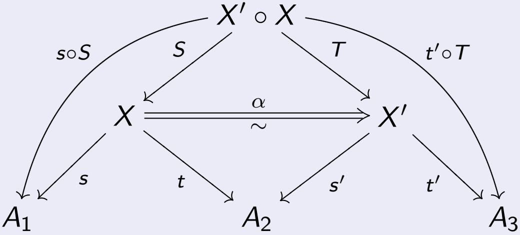
A *span* in a (n-)category $\mathbf{C}$ is a diagram:



The bicategory $Span_2(\mathbf{Gpd})$ (similar for any 2-category with weak pullbacks) has:

Definition

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- Composition defined by *weak* pullback:



- **2-Morphisms** : isomorphism classes of *spans of* span maps
- monoidal structure from the product in **Gpd**, monoidal unit 1

(The category $Span(\mathbf{Gpd})$ takes spans up to span-isomorphism)

Definition

$$D : Span(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$$

with $D(G) = \mathbb{C}(\underline{G})$,

$$D(X)(f)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(x)} [f(s(x))]$$

This amounts to multiplication by a matrix $D(X)$ with

$$D(X)_{([a],[b])} = \left| (s,t)^{-1}(a,b) \right|$$

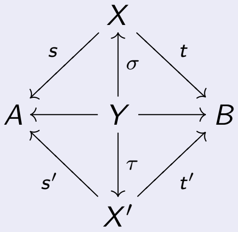
using *groupoid cardinality* (which can be interpreted as an inner product in a canonical way).

(*Note*: Compare gpd. cardinality to role of Euler characteristic in geom. representation theory.)

D is a "quotient-style" decategorification map.

Definition (Part 2)

The **2-morphisms** of $Span_2(\mathbf{Gpd})$ are (iso. classes of) spans of *span maps*:



Composition is by weak pullback taken up to isomorphism.

(Often one just uses span maps: here, we want 2-morphisms as well as morphisms to have *adjoints*, so much use these.)

Definition

2Vect is the 2-category of *Kapranov-Voevodsky* 2-vector spaces, which consists of:

- Objects: **Kapranov–Voevodsky 2-vector spaces**: $\mathbb{C}$-linear finitely semisimple additive category (one generated by simple objects x, where $\hom(x, x) \cong \mathbb{C}$).
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence additive) functor.
- 2-Morphisms: Natural transformations between 2-linear maps

Note: **2Vect** is a monoidal 2-category with the Deligne product and unit **Vect**.

Theorem (KV)

Any KV 2-vector space is equivalent to $\mathbf{Vect}^k$ *for some* k*. Any 2-linear map is then naturally isomorphic to one given by a matrix of vector spaces (and matrix multiplication using* $\otimes$ *and* $\oplus$*). Any natural transformation of 2-linear maps is then given by a matrix of componentwise linear maps.*

Lemma

If $\mathbf{B}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{B}) = [\mathbf{B}, \mathbf{Vect}]$ *is a KV 2-vector space.*

Note: If the automorphism groups of (isomorphism classes of) objects of $\mathbf{B}$ are $B_1, \ldots, B_n$, then we have

$$[\mathbf{B}, \mathbf{Vect}] \cong \prod_j \mathbf{Rep(B_j)}$$

So the "basis elements" (simple objects) in $[\mathbf{B}, \mathbf{Vect}]$ are labeled by $([b], V)$, where $[b] \in \underline{\mathbf{B}}$ and V an irreducible rep of $Aut(b)$.

Theorem

If $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \to \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \to \Lambda(\mathbf{X})$$

namely composition with f*, with* $f^*F = F \circ f$ *and*

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{B})$$

called "pushforward along f*". Furthermore,* f_* *is the two-sided adjoint to* f^* *(i.e. both left-adjoint and right-adjoint).*

In fact, the adjoint map f_* acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x) \cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is the left adjoint. But there is also a right adjoint:

$$f_!(F)(b) \cong \bigoplus_{[x] | f(x) \cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

The *Nakayama isomorphism* is a canonical isomorphism between these (in particular: it defines an isomorphism even over base rings other than $\mathbb{C}$). It gives maps:

$$N_{(f,F,b)} : f_!(F)(b) \longrightarrow f_*(F)(b)$$

given by the *exterior trace map* in each factor of the sum (which uses a modified group average):

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g \in Aut(b)} g \otimes \phi_x(g^{-1})$$

Under this identification, the left and right adjoints are isomorphic. By composing units/counits with N, we get that f^* and f_* are ambidextrous adjoints.
(Note: In general, $Span_2(\mathbf{C})$ will be the universal 2-category for which morphisms in $\mathbf{C}$ have ambidextrous adjoints. We want to preserve this property.)

Call the adjunctions in which f_* is left or right adjoint to f^* the *left and right adjunctions* respectively. We want to use the counit for the left adjunction, which is the evaluation map:

$$\begin{array}{rcl} \eta_R(G)(x) : G(x) & \rightarrow & \bigoplus_{y|f(y)\cong x} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(y)], G(x)) \\ v & \mapsto & \bigoplus_{y|f(y)\cong x} (g \mapsto g(v)) \end{array}$$

and the unit for the right adjunction, which just uses the action:

$$\begin{array}{rcl} \epsilon_L(G)(x) : \bigoplus_{[y]|f(y)\cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} f^*G(x) & \rightarrow & G(x) \\ \bigoplus_{[y]|f(y)\cong x} g_y \otimes v & \mapsto & \sum_{[y]|f(y)\cong x} f(g_y)v \end{array}$$

Definition

Define the 2-functor Λ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathbf{a}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, we get that $\Lambda(X, s, t)$ is represented by the matrix with coefficients:

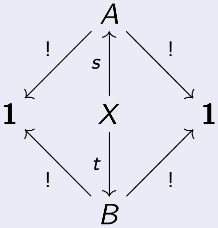
$$\Lambda(X, s, t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$
$$\simeq \bigoplus_{\underline{[x]\in(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

This is an intertwiner space, given by the *analog* of an inner product. The 2-morphisms give linear maps between intertwiner spaces, which can also be interpreted as a "pull-push" operation.

In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

Theorem

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*



where $\mathbf{1}$ *is the (terminal) groupoid with one object and one morphism,* Λ *on 2-morphisms is just the degroupoidification functor* D*.*

The groupoid cardinality comes from the modified group average in N.

2-Linearized Physics

"Physical" applications arise because groupoids provide a good way of thinking about local symmetry, generalizing the *action groupoid* $S /\!\!/ G$ associated to a G-action on S.

In a span $A \leftarrow X \rightarrow B$, the groupoid X will represent a space of *histories*; s and t pick the starting and terminating *configuration* in spaces A and B.

This setup is how we "do physics in" the monoidal (2-)category $Span_2(\mathbf{Gpd})$. The functors D and Λ will give a description of physics in **Vect** (really, **Hilb** since there is a canonical inner product), and **2Vect** respectively (ditto).

The span $\mathbf{Vect} \leftarrow Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$ provides a way to "categorify quantum mechanics".

Example

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Vect}$$

where $\mathbf{nCob_2}$ has

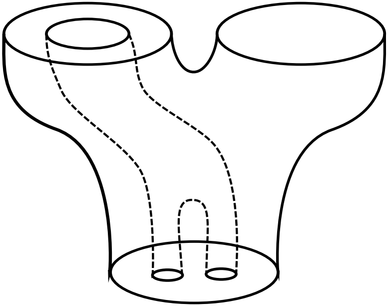
- **Objects**: $(n-2)$-dimensional manifolds
- **Morphisms**: $(n-1)$-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)
- **2-Morphisms**: n-dimensional cobordisms with corners

One construction uses *gauge theory*, for gauge group G (here a finite group). Given M, the groupoid $\mathcal{A}_0(M,G) = hom(\pi_1(M), G) /\!\!/ G$ has:

- **Objects**: Flat connections on M
- **Morphisms** Gauge transformations

Then $\mathcal{A}_0(-,G) : \mathbf{nCob_2} \to Span_2(\mathbf{Gpd})$, and there is an ETQFT $Z_G = \Lambda \circ \mathcal{A}_0(-,G)$.
It happens to give the *Dijkgraaf-Witten model* when $n = 3$.

This relies on the fact that cobordisms in $\mathbf{nCob_2}$ can be transformed into products of cospans:

$\mathbf{nCob_2}$	$Span^2(Top)$
	$\begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A\coprod D) & \xleftarrow{i'_A \otimes i_D} & S^1 \coprod S^1 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle i_2} \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle i_1} \\ S^1 \coprod S^1 & \xrightarrow[i_2]{} & Y & \xleftarrow[i_1]{} & S^1 \end{array}$

Then $\mathcal{A}_0(-, G)$ maps these into $Span^2(\mathbf{Gpd})$.

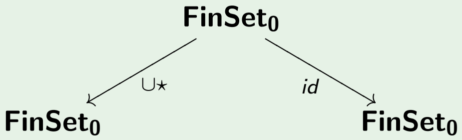
Example

In the case where $\mathbf{A} = \mathbf{B} = \mathbf{FinSet_0}$ (equivalently, the symmetric groupoid $\coprod_{n \geq 0} \Sigma_n$ - note no longer finite), we find

$$D(\mathbf{FinSet_0}) = \mathbb{C}[[t]]$$

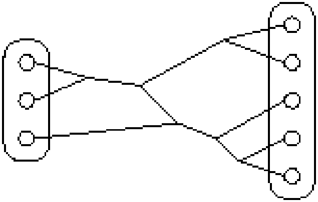
where t^n marks the basis element for object $[n]$. This gets a canonical inner product and can be treated as the Hilbert space for the *quantum harmonic oscillator* ("Fock Space").

The operators $\mathbf{a} = \partial_t$ and $\mathbf{a}^\dagger = M_t$, generate the *Weyl algebra* of operators for the QHO. These are given under D by the span A:



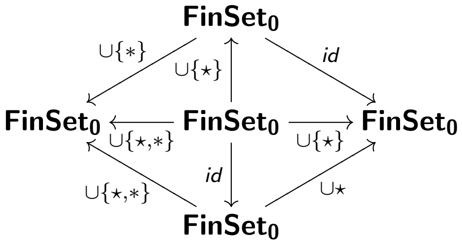
and its dual $A^\dagger$. Composites of these give a categorification of operators explicitly in terms of *Feynman diagrams*.

Such composites are described in terms of groupoids whose objects look like this:



The source and target maps for the span pick the set of start and end points. The morphisms of the groupoid are graph symmetries. Degroupoidification D calculates operators which (after small modification involving $U(1)$-labels) agree with the usual Feynman rules for calculating amplitudes.

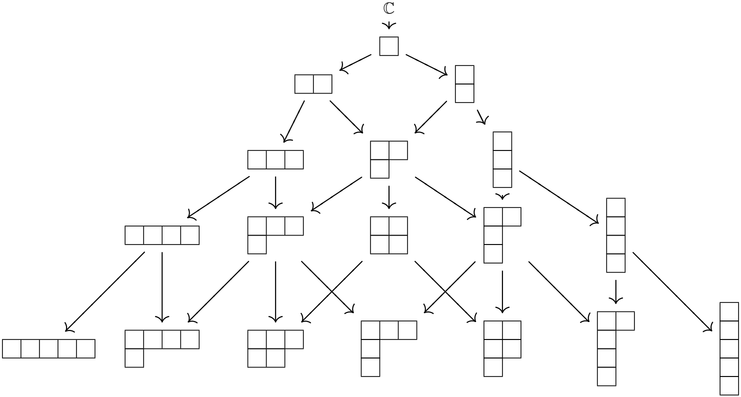
An ongoing project (with Jamie Vicary) is to study the 2-categorical version of this picture. There are analogs of creation and annihilation operators in other *hom*-categories than $hom(1,1)$:



This is a 2-morphism $\alpha_A : A \longrightarrow AAA^\dagger$ creates a "creation/annihilation pair" at the 1-morphism level.

Composites of these act as *rewrite rules* on the Feynman diagrams like those seen previously (now with "boundary" maps).

The image of this picture under Λ involves representation theory of the symmetric groups as $\Lambda(\mathbf{FinSet_0}) \cong \prod_n Rep(\Sigma_n)$, and gives rise to "paraparticle statistics":



Toward Real QFT

Both the QHO and TQFT are "baby" models of real QFT, which is much harder.

One ingredient: the construction for Λ can be extended to measure-groupoids (e.g. derived from compact Lie groups w/ Haar measure), using:

- **Vect** $\mapsto$ **Hilb** (ambiadjoint uses double-dual isomorphism)
- *Rep*(**B**) $\mapsto$ Category of reps of von Neumann algebra associated to **B**
- 2-linear maps represented by *Hilbert bimodules*
- Direct sum $\mapsto$ direct integral
- Groupoid cardinality $\mapsto$ volume of groupoid (c.f. Weinstein)

This relates to a conjecture of Baez et. al. that *infinite-dimensional 2-Hilbert spaces* are equivalent to representation categories for v.N.-algebras.