# Groupoidification and Khovanov's Categorification of the Heisenberg Algebra

(Joint work with Jamie Vicary)

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- set-based structures $\Rightarrow$ category-based structures
- not systematic: any inverse to some *decategorification* process, such as:
  - Degroupoidification (Baez-Dolan): a functor $D : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{Hilb}$
  - Khovanov-Lauda: $C \mapsto K_0(C)$, the Grothendieck ring (used for algebraic categorification of quantum groups)
- Goal: describe an example in which these two approaches are related

The one-variable **Heisenberg algebra** is an algebra $H$ given by two generators $\mathbf{a}$ ("annihilation") and $\mathbf{a}^\dagger$ "creation"), satisfying the *canonical commutation relation*:

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1 \tag{1}$$

The general Heisenberg algebra has generators $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ for each $i = 1, \dots, n, \dots$.

There is only one nontrivial, irreducible representation (which is faithful) of the algbera, on **Fock space**, $H \mapsto Aut(\mathcal{F})$, where:

$$\mathcal{F} = \mathbb{C}[\![z]\!]$$

(The space of (formal) power series in $z$).

In this representation, the algebra is generated by:

$$\mathbf{a}f(z) = \partial_z f(z) \tag{2}$$

and

$$\mathbf{a}^\dagger f(z) = zf(z) \tag{3}$$

The commutation relation holds for $\mathbf{a}$ and $\mathbf{a}^\dagger$, since:

$$\partial_z(zf(z)) = z\partial_z f(z) + f(z)$$

If we define an inner product on $\mathcal{F}$ where $\{z^n\}$ is an orthogonal basis such that

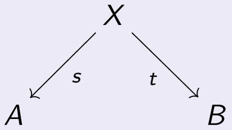
$$\langle z^n, z^n \rangle = \frac{1}{n!}$$

then $\mathbf{a}^\dagger$ is the adjoint of $\mathbf{a}$.

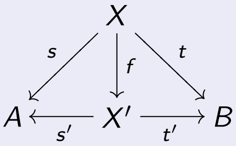
## Definition (Part 1)

The (monoidal) bicategory *Span*(**Gpd**) has:

- **Objects** (Essentially finite/countable) groupoids
- **Morphisms** Spans of groupoids:

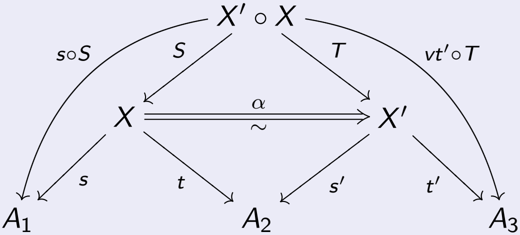



- **2-Morphisms**: Span maps $f$:



## Definition (Part 2)

- Composition Span(**Gpd**) is defined by *weak* pullback:




- Span(**Gpd**) has monoidal structure determined by the fact that **Gpd** is Cartesian, so $A \otimes B \in$ Span(**Gpd**) is $A \times B \in$ **Gpd**

Write $\text{Span}_1$(**Gpd**) for the homotopy 1-category, whose morphisms are iso. classes of 1-morphisms in Span(**Gpd**).

## Definition (Baez-Dolan)

The **degroupoidification functor** acts on

$$D : (\mathrm{Span}_1(\mathbf{Gpd})) \rightarrow \mathbf{Hilb}$$

assigns to a groupoid $G$

$$D(G) = \mathbb{C}(\underline{G})$$

which is given an inner product where

$$\langle \delta_a, \delta_b \rangle = \frac{\delta_{a,b}}{\# Aut(a)}$$

To a span $(X, s, t)$, $D$ assigns the linear map

$$t_* \circ s^* : D(A) \rightarrow D(B)$$

where

$$s^* : \mathbb{C}(\underline{A}) \rightarrow \mathbb{C}(\underline{X})$$

acts by composition with $s$, and $t_*$ is the $\langle \cdot, \cdot \rangle$-adjoint of $t^*$.

This amounts to a linear operator:

$$D(X)(f)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \operatorname{Aut}(b)}{\# \operatorname{Aut}(x)} [f(s(x))]$$

which is represented by the matrix

$$D(X)_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

using *groupoid cardinality*.

Physically, $X$ will represent a groupoid of *histories* leading a system $A$ to the system $B$. Maps $s$ and $t$ pick the starting and terminating *configurations* in $A$ and $B$ for a given history.

**Definition**

A **state** for an object $A$ in a monoidal category is a morphism from the monoidal unit, $\psi : I \rightarrow A$.

In **Hilb**, this determines a vector by $\psi : \mathbb{C} \rightarrow H$. In *Span*(**Gpd**), the unit is **1**, the terminal groupoid, so this is determined by:

$$S \xrightarrow{\Psi} A$$

where $S$ is a groupoid, over $A$.

The Heisenberg algebra acting on Fock space describes the "quantum harmonic oscillator", one of the simplest quantum mechanical systems.

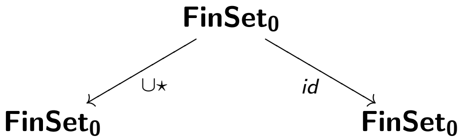
## The Heisenberg Algebra Again

Consider the groupoid $\mathbf{FinSet_0}$ (equivalently, the symmetric groupoid $\coprod_{n\geq 0}\mathcal{S}_n$), we find

$$D(\mathbf{FinSet_0}) = \mathbb{C}[\![z]\!]$$

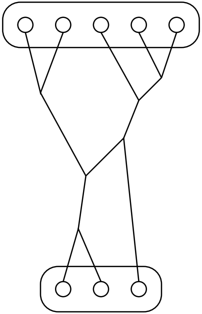
where $z^n$ marks the basis element $\delta_{[n]}$, with the correct inner product for *Fock space*.

Consider the span $A$:




and its dual $A^\dagger$. These generate a subcategory $\mathbf{h}$ of $End_{\mathrm{Span}(\mathbf{Gpd})}(\mathbf{FinSet_0})$. Then $D(A) = \mathbf{a} = \partial_t$ and $D(A^\dagger = \mathbf{a}^\dagger = z$. So $D(\mathbf{h}) \cong H$, the Heisenberg algebra.

Such composites are described in terms of groupoids whose objects are *Feynman diagrams*:



The source and target maps for the span pick the set of start and end points. The morphisms of the groupoid are graph symmetries. Degroupoidification $D$ calculates operators which (after small modification involving $U(1)$-labels) agree with the usual Feynman rules for calculating amplitudes for the quantum harmonic oscillator.

## The Fock Monad

The Fock space $\mathcal{F}$ comes from a general construction

$$F(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes_s n}$$

where $\mathcal{F} = F(\mathbb{C})$.

This can be defined for any symmetric †-monoidal category $\mathbf{C}$ with †-biproducts. This $F$ is a monad which arises from an adjunction as $F = R \circ Q$:

$$\mathbf{C} \underset{R}{\overset{Q}{\rightleftarrows}} \mathbf{C}_\times$$

where $\mathbf{C}_\times$ is the category of cocommutative comonoid objects in $\mathbf{C}$, and $R$ is the forgetful functor.

The structure of the operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ arises from the fact that $F(V)$ naturally gets a bialgebra structure for any object $V \in \mathbf{C}$.

The choice of the groupoid $\mathbf{FinSet_0}$ is made for similar reasons. There is a similar situation for groupoids:

$$\mathbf{Gpd} \underset{R}{\overset{Q}{\rightleftarrows}} \mathbf{Gpd}_{\times}$$

Then we can take the free symmetric monoidal category on a groupoid:

$$F_s(G) = \coprod_{n \in \mathbb{N}} S_n \ltimes G^n$$

which is a groupoid with:

- **Objects**: $n$-tuples $g_1 \otimes \cdots \otimes g_n \in G^n$ for some $n$
- **Morphisms** $(\phi, (f_1, \ldots, f_n))$ with $\phi \in \mathcal{S}_n$ and $f_i : g_i \to g'_{\phi(i)}$

In particular, $F_s(\mathbf{1}) \simeq \mathbf{FinSet_0}$.
We have $D \circ F_s = F \circ D$, so that $D(\mathbf{FinSet_0})$ is Fock space.

**Khovanov's Categorification**

The categorification of the (multivariable!) Heisenberg algebra is an example of the Khovanov-Lauda approach to categorifying Lie algebras, quantum groups, etc.

**Definition**

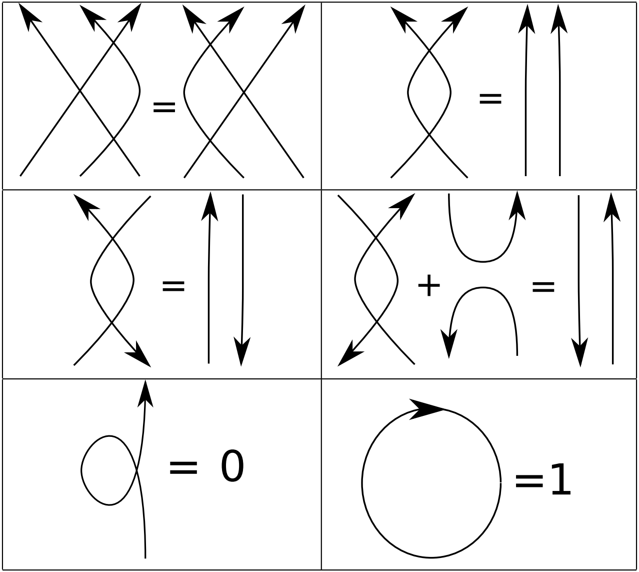
There is a monoidal category **H** with

- Objects: generated by points labelled $Q_+$ ("up") and $Q_-$ ("down")
- Morphisms: linear combinations of (string diagrams, agreeing with orientations at endpoints, taken up to isotopy and certain local moves):

The monoidal category $\mathbf{H}'$ is the *Karoubi envelope* $\mathbf{H} = Kar(\mathbf{H}')$.

(The Karoubi envelope $\mathbf{H}'$ makes all idempotents split. It includes symmetric and antisymmetric powers of the objects, $S^n_\pm = S^n(Q_\pm)$ and $\bigwedge_\pm = \bigwedge^n(Q_\pm)$, respectively.)

Local Moves for morphisms of **H**:



Commutation relations become specified isomorphisms, which are described by such diagrams. For example:

$$S^n_s \otimes \Lambda^m_+ \cong (\Lambda^m_+ \otimes S^n_-) \oplus (\Lambda^{m-1}_+ \otimes S^{n-1}_-) \tag{4}$$

**Proposition (Khovanov)**

*There is a surjective map $K_0(\mathbf{H}') \to H_+$ (onto the positive integer form of the Heisenberg algebra).*

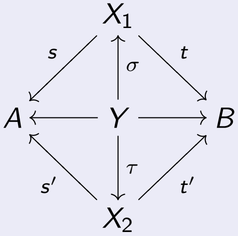
(Khovanov conjectures it is an isomorphism.)

**Question**: How is this related to groupoidification?

There is a (monoidal) 3-category $Span_2(\mathbf{Gpd})$ which allows all the 2-cells from **Gpd** to have adjoints...

## Definition (Part 3)

The **2-morphisms** of $Span_2(\mathbf{Gpd})$ are spans of *span maps*, commuting up to 2-cells of **Gpd**:




These are taken up to isomorphism. Composition is by weak pullback as for 1-morphisms.

There are "horizontal and vertical duals" for each 2-morphism.

## Ambiadjunctions

- For Cartesian **C**, *Span***C** is the *universal* 2-category containing **C**, for which every morphism in **C** has a (two-sided) adjoint.
- In fact, that *Span*(**C**) is a †-monoidal, †-abelian category. This is useful to describe quantum physics. (See Abramsky and Coecke, Vicary).
- Span(**Gpd**) is a universal 3-category containing **Gpd** such that every morphism contains a two-sided adjoint

The span $F : A \to B$ given as

$$\begin{array}{ccc} & X & \\ {}^{s}\swarrow & & \searrow^{t} \\ A & & B \end{array}$$

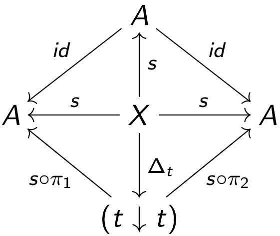
has ambiadjoint $F^\dagger : B \to A$ found by reversing orientation:

$$\begin{array}{ccc} & X & \\ {}^{t}\swarrow & & \searrow^{s} \\ B & & A \end{array} \tag{5}$$

To fully specify the ambiadjunction, however, we need four unit and counit 2-morphisms:

$$\eta_L : Id_A \to F \circ F^\dagger$$
$$\eta_R : Id_B \to F^\dagger \circ F$$
$$\epsilon_L : F^\dagger \circ F \to Id_B$$
$$\epsilon_R : F \circ F^\dagger \to Id_A$$

We have $\eta_L = \epsilon_R^{co}$:



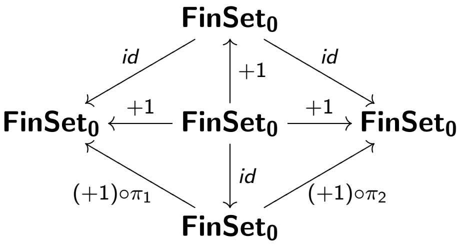

- $(t \downarrow t)$ is the comma category whose objects are $(x, f, x')$ with $f : t(x) \rightarrow t(x')$, and whose morphisms are commuting squares
- $\Delta_t : X \rightarrow (t \downarrow t)$ takes objects $x \mapsto (x, id_{t(x)}, x)$ and morphisms $g \mapsto (g, g)$

And similarly for $\eta_R = \epsilon_L^{co}$.
These satisfy the usual adjunction properties:

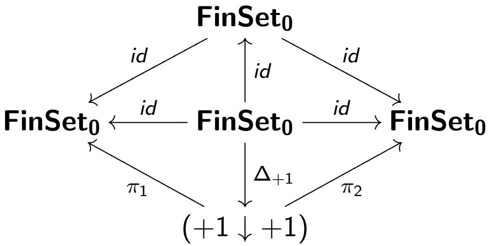
$$(Id \circ \eta_L) \cdot (\epsilon_L \circ Id) = Id$$

$$(\eta_R \circ Id) \cdot (Id \circ \epsilon_R) = Id$$

Take the case where $F = A$, the groupoidified annihilation operator. Then the left unit $\eta_L : Id_{\mathbf{FinSet_0}} \Rightarrow A \circ A^{\dagger}$ is (equivalent to):




And the right unit $\eta_R : Id_{\mathbf{FinSet_0}} \Rightarrow A^{\dagger} \circ A$ is:



Where $(+1 \downarrow +1)$ can be described up to equivalence by:

- **Objects**: $(S_1, \phi, S_2)$, where $\phi : (S_1 \sqcup \star) \to (S_2 \sqcup \star)$ is an isomorphism
- **Morphisms**: Pairs $(f_1, f_2)$, $f_i : S_i \to S_i'$ giving commuting squares:

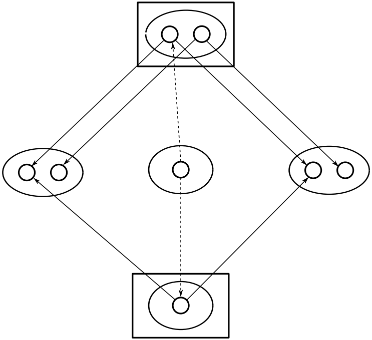
$$\begin{array}{ccc} (+1)(S_1) & \xrightarrow{\phi} & (+1)(S_2) \\ {\scriptstyle (+1)(f_1)}\downarrow & & \downarrow{\scriptstyle (+1)(f_2)} \\ (+1)(S_1') & \xrightarrow{\phi'} & (+1)(S_2) \end{array}$$

Up to equivalence, this amounts to:

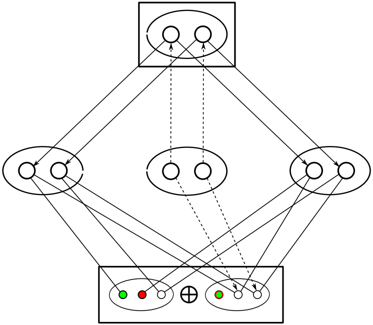
- **Objects**: $(n, \phi, n)$, where $\phi \in \mathcal{S}_{n+1}$
- **Morphisms**: $(\pi_1, \pi_2) \in \mathcal{S}_n^2$ such that $\phi' \circ \pi_1 = \pi_2 \circ \phi$

Note that all these constructions depend only on the groupoids *up to equivalence* (in fact, they are constructions involving stacks.)

Internally, these look like:



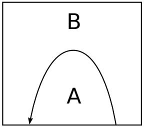
$\eta_L : Id_{\mathbf{FinSet_0}} \Rightarrow A \circ A^\dagger$



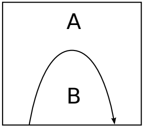
$\eta_R : Id_{\mathbf{FinSet_0}} \Rightarrow A^\dagger \circ A$

This also shows why $A^\dagger \circ A \cong A \circ A^\dagger \oplus Id$, (groupoidifies the relation $[\mathbf{a}, \mathbf{a}^\dagger] = 1$): because "add-then-remove" has one more possibility than "remove-then-add".

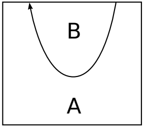
The units and counits for any ambiadjunction of $F : A \to B$ can be represented graphically:



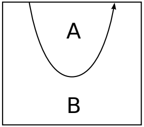

$\eta_L : Id_B \Rightarrow F \circ F^\dagger$




$\eta_R : Id_A \Rightarrow F^\dagger \circ F$




$\epsilon_L : F^\dagger \circ F \Rightarrow Id_A$




$\epsilon_R : F \circ F^\dagger \Rightarrow Id_B$

## Correspondence

**Theorem**

*There is a functor from Khovanov's* $\mathbf{H}'$

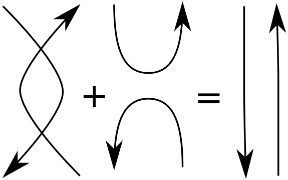
$$I : \mathbf{H}' \to \mathbf{W} \subset End_{\mathrm{Span}(\mathbf{Gpd})}(\mathbf{FinSet_0}) \tag{6}$$

*given by the correspondences:*

| $\mathbf{H}'$ | $\mathbf{W} \subset \mathrm{Span}(\mathbf{Gpd})$ |
|---|---|
| $\mathbf{H}'$ | $\mathbf{W}$ |
| $\bullet$ | $\mathbf{FinSet_0}$ |
| $Q_-,\ Q_+$ | $A,\ A^\dagger$ |
| $\downarrow, \uparrow$ | $Id_A,\ Id_{A^\dagger}$ |
| $\otimes$ | $\circ$ |
| $\cap, \cup$ | $\eta,\ \epsilon$ |
| *crossings* | $S_n$ *action* |

(Proof: check **W** satisfies all the local moves..)

Then there are combinatorial interpretations of pictures like this:



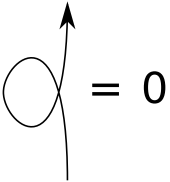

Namely: "add-then-remove" has one more possibility than "remove-then-add", since

- The RHS shows the identity on $A^\dagger \circ A$
- First term on LHS swaps the order to give $A \circ A^\dagger$ (selects the case "remove a different element from that added"
- Second term selects the case "remove the same element added" (otherwise the counit is zero)

**Note**: This is also one of the equations for a *biproduct*:

$$A \circ A^\dagger \cong A^\dagger \circ A \oplus Id \tag{7}$$

Similarly, there is an interpretation of:



Namely, that a certain sequence of changing processes cannot be done:

- Add new element $x$ into a set
- *(insert add-remove pair)*
- Add new element $x$, then $y$, then remove $y$
- *(swap adding $x$ and $y$)*
- Add $y$, then $x$, then remove $y$
- *(cancel add-x-remove-y pair: IMPOSSIBLE)*
- Add $y$

Khovanov proves the main result about $\mathbf{H}'$ using a category based on bimodules representing restriction and induction functors. This shows up in Span(**Gpd**) by:

### Theorem

*There is an ambiadjunction-preserving 2-functor ("2-linearization"):*

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

Where, recall:

### Definition

**2Vect** is the 2-category of 2-vector spaces, which consists of:

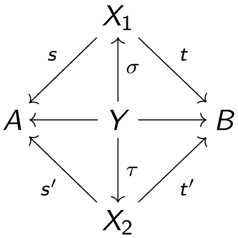
- Objects: $\mathbb{C}$-linear abelian category, generated by simple objects
- Morphisms: **2-linear maps**: $\mathbb{C}$-linear (hence abelian) functor.
- 2-Morphisms: Natural transformations

## Definition

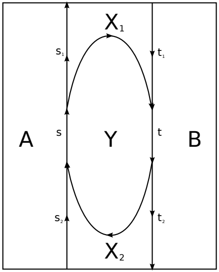
Define the 2-functor $\Lambda$ as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Morphisms $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathbf{A}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Morphisms: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \to (t')_* \circ (s')^*$

This is summarized graphically as:




$\Rightarrow$



The map $N$ is a special isomorphism between the left and right adjoints of $s^*$ or $t^*$.
For any homomorphism of groupoids $f$, the LEFT adjoint map of $f^*$, called $f_*$, acts by:

$$f_*(F)(b) \cong \bigoplus_{f(x)\cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a (left) *Kan extension* of the functor $F$ along $f$.

There is also a RIGHT adjoint (right Kan extension):

$$f_!(F)(b) \cong \bigoplus_{[x]|f(x)\cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

We want to represent this by tensoring with a bimodule as with $f_*$.

There is the canonical *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \to f_*(F)(b)$$

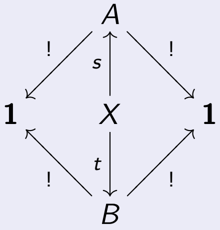
given by the *exterior trace map* (which uses a modified group average in each factor):

$$N : \bigoplus_{[x] | f(x) \cong b} \phi_x \mapsto \bigoplus_{[x] | f(x) \cong b} \frac{1}{\# Aut(x)} \sum_{g \in Aut(b)} g \otimes \phi_x(g^{-1})$$

Under this identification we get that $f^*$ and $f_*$ are ambidextrous adjoints.

### Theorem

*Restricting to* $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1},\mathbf{1})$*:*




$\Lambda$ *on 2-morphisms is just the degroupoidification functor* $D$*.*

Since any $Hom(A,B)$ has a map to $Hom(\mathbf{1},\mathbf{1}$ by composing with the unique maps $A,B \rightarrow \mathbf{1}$, the original groupoidification is recovered from the image of $\eta_L$ and its dual.

There is a pseudomonad

$$F_v : \mathbf{2Vect} \rightarrow \mathbf{2Vect}$$

which assigns the free symmetric monoidal 2-vector space for an object $X$:

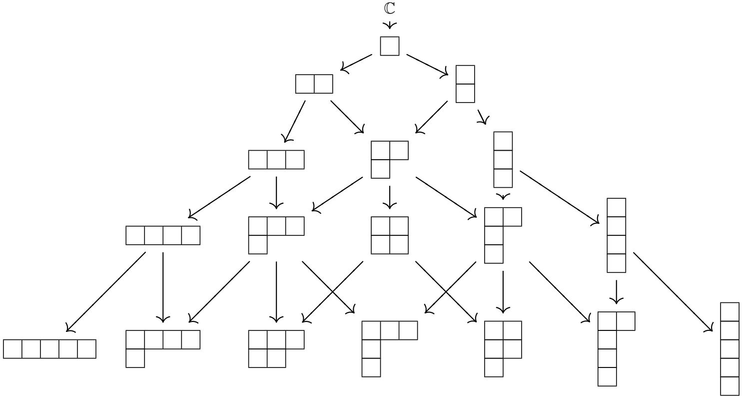
$$F_v(X) = \oplus_{n \in \mathbb{N}} X^{\otimes_s n} \tag{8}$$

The symmetric tensor product $X \otimes_s X$ is a pseudo-limit, using an equifier 2-cell for the diagram:

$$X \otimes X \underset{id_{X \otimes X}}{\overset{\tau_{X,X}}{\rightrightarrows}} X \otimes X \tag{9}$$

(This gives an action of the permutation group on any object of $X^{\otimes_s n}$.)
We have $\Lambda \circ F_s \cong F_v \circ \Lambda$, $F_v(\mathbf{Vect}) \simeq \Lambda(\mathbf{FinSet_0}) = Rep(\mathbf{FinSet_0})$.

The 2-vectorial "Fock space" is $\Lambda(\mathbf{FinSet_0}) \cong \prod_n Rep(\Sigma_n)$. $\Lambda(A)$ and $\Lambda(A^\dagger) = \oplus_n(- \otimes \mathbb{C}^n)$ give representations counting paths in this lattice:



To get Khovanov's symmetric and antisymmetric products, $S^n_\pm$ and $\bigwedge^n_\pm$, we use:

### Theorem

*Given any monoidal category* $\mathbf{C}'$ *for which we have an inclusion* $\bullet /\!\!/ \mathbf{C} \subset \mathbf{2Vect}$, *there is also an inclusion of the Karoubi envelope* $\mathbf{C} = Kar(\mathbf{C}')$ *such that the following commutes:*

$$\begin{array}{ccc} \bullet /\!\!/ \mathbf{C}' & \xrightarrow{i'} & \mathbf{2Vect} \\ {\scriptstyle Kar}\downarrow & \nearrow_{i} & \\ \bullet /\!\!/ \mathbf{C} & & \end{array} \tag{10}$$

Then we apply this to:

$$\mathcal{H}' \xrightarrow{I} \mathbf{W} \xrightarrow{\Lambda} \mathbf{2Vect} \tag{11}$$

## References

Our Work:

- Jeff Morton: arXiv:math.CT/0611930, arXiv:0810.2361
- Jamie Vicary: arxiv:0706.0711

Khovanov's Categorification (and KL program):

- Khovanov: arXiv:1009.3295
- Khovanov-Lauda: arXiv:0803.4121, arXiv:0804.2080