Groupoid Representation Theory

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Introduction

- We've seen that *stacks* are presented by groupoids
- If $\mathbf{G}$ and $\mathbf{G}'$ are "Morita equivalent", they present the same stack
- Morita equivalence also amounts to saying that $Rep(\mathbf{G})$ and $Rep(\mathbf{G}')$ are equivalent as categories
- This is related to equivalence of representation categories of groupoid algebras
- Representation theory of groupoids is also important in mathematical physics (AQFT, ETQFT)

Groupoids

Definition

A **groupoid G** (in **C**) is a category (in **C**) in which all morphisms are invertible. That is, there are **C**-objects M (of objects) and G (of morphisms) together with structure maps (**C**-morphisms):

$$s, t : G \longrightarrow M \tag{1}$$

$$i : M \longrightarrow G \tag{2}$$

$$\circ : G \times_M G \longrightarrow G \tag{3}$$

$$(-)^{-1} : G \longrightarrow G \tag{4}$$

satisfying the usual properties. When $\mathbf{C} = \mathbf{Diff}$ (with some extra conditions), this is a **Lie groupoid**.

As a shorthand, we often write **G** as $s, t : G \longrightarrow M$.

Groupoid Actions

Definition

If $\mathbf{G} = s, t : G \to M$ is a Lie groupoid, and $\tau : X \to M$ is smooth, a left $\mathbf{G}$-action on X is a smooth map

$$\triangleright : G \times_{M,s,\tau} X \to X \tag{5}$$

This map takes (g, x) to $g \triangleright x$ such that:

$$\tau(g \triangleright x) = t(g) \tag{6}$$

identities act by:

$$1_m \triangleright x = x \tag{7}$$

and

$$g \triangleright (g' \triangleright (x)) = (gg') \triangleright (x) \tag{8}$$

whenever these are defined.
Right actions $\triangleleft : X \times_{M,\tau,t} G \to X$ are definied similarly.

Representations on Vector Bundles

A representation of a group is an action on a vector space V. This amounts to a homomorphism into $End(V)$, the group of endomorphisms of a vector space V.

A representation of a groupoid is an action on a vector bundle $E \rightarrow M$. The *frame groupoid* generalizes the group $End(V)$:

Definition

Given a vector bundle $q : E \rightarrow M$, the **frame groupoid** $\mathbf{\Phi}(E) = s, t : \Phi(E) \rightarrow M$ consists of $\Phi(E)$, the set of all vector space isomorphisms $\xi : E_x \rightarrow E_y$ over all $(x, y) \in M^2$, with the obvious structure maps.

So a representation, i.e. an action on a vector bundle, amounts to the following:

Definition

A representation of a Lie groupoid $s, t : G \longrightarrow M$ on a vector bundle $q : E \longrightarrow M$ is a smooth homomorphism (i.e. functor):

$$\rho : \mathbf{G} \longrightarrow \mathbf{\Phi}(E) \tag{9}$$

of Lie groupoids over M.

A representation ρ necessarily gives a functor $R : \mathbf{G} \longrightarrow \mathbf{Vect}$ with $R(x) = E_x$, the fibre at each $x \in M$, and an isomorphism $R(g)$ for each $g : x \longrightarrow y$. (But not all functors are *smooth* representations).

Morphisms of Representations

Definition

Rep(**G**), the **category of representations** of **G**, has

- *Objects*: Representations of **G**
- *Morphisms*: Intertwiners from ρ to ρ' are a bundle morphisms $i : E \longrightarrow E$ over M so that $\rho'(g) \circ i = i \circ \rho$

Note: treating representations as functors into **Vect**, an intertwiner is a natural transformation between such functors, implemented by bundle morphisms i in the relevant category (e.g. **Diff**.

Morita Equivalence

Definition

Two categories (say groupoids $\mathbf{G}$ and $\mathbf{G}'$) are **equivalent** if there are functors $f : \mathbf{G} \rightarrow \mathbf{G}'$ and $g : \mathbf{G}' \rightarrow \mathbf{G}$ with $g \circ f \simeq \mathrm{Id}_{\mathbf{G}}$ and $f \circ g \simeq \mathrm{Id}_{\mathbf{G}'}$.

Applying this to categories of representations gives another notion of equivalence:

Definition

Two groupoids are **Morita equivalent** if their categories of representations are equivalent (as symmetric monoidal categories).

This is a quite general idea of equivalence which can be applied to anything with "representations" (or more generally modules): groupoids, rings, algebras, etc.

Morita Morphisms

We want to know conditions when two groupoids are Morita equivalent. One condition involves the following:

Definition

A **Morita morphism** from $\mathbf{G}$ to $\mathbf{G}'$ is a pair of morphisms:

$$\mathbf{G} \xleftarrow{\;f\;} \mathbf{X} \xrightarrow{\;g\;} \mathbf{G}' \tag{10}$$

where both f and g are fibrations - i.e. satisfy the homotopy lifting property.

That is, if a functor into $\mathbf{G}$ can be lifted to $\mathbf{X}$, so can a homotopy of this functor. (Note: for **Set**-groupoids, any f is a fibration.)

Definition

A map $f : \mathbf{H} \to \mathbf{G}$ of topological/Lie groupoids is an **essential equivalence** if:

- The map $t \circ \pi_1 : G \times_M N \to M$ is surjective (and a submersion in the Lie case)
- The square

$$\begin{array}{ccc} H & \xrightarrow{f_1} & G \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ N \times N & \xrightarrow[(f_0,f_0)]{} & M \times M \end{array} \tag{11}$$

  is a pullback of spaces.

This amounts to an equivalence of categories (a full, faithful, essentially surjective functor). If there is such an f, then $Rep(\mathbf{H}) \simeq Rep(\mathbf{G})$.

Define an equivalence relation on topological groupoids so that

$$\mathbf{H} \sim \mathbf{G} \tag{12}$$

whenever there is an essential equivalence between **H** and **G**.

Theorem

Two topological groupoids **G** *and* $\mathbf{G}'$ *are equivalent in the above equivalence relation if and only if there is a Morita morphism*

$$\mathbf{G} \xleftarrow{\;f\;} \mathbf{X} \xrightarrow{\;g\;} \mathbf{G}' \tag{13}$$

such that both f and g are essential equivalences.

It follows that when such a Morita morphism exists, **G** and $\mathbf{G}'$ are Morita equivalent. Proving this for *Lie* groupoids is harder. This uses the technology of bibundles...

Bibundles

For Lie groupoids, the above constructions are trickier, since in general, pullbacks (needed to compose Morita morphisms for the characterization of $\sim$) do not exist in **Diff** unless certain conditions are satisfied. A different approach is usual:

Definition

A **left** (right) **G-bundle** E over X (equipped with $\tau : X \to M$) is a left (right) **G**-action on E, and a **G**-invariant map

$$\pi : E \to M \tag{14}$$

A **G**-**H**-bibundle E is a left **G**-bundle and a right **H**-bundle.

Bibundles can encode ordinary maps:

Definition

If $f : \mathbf{G} \to \mathbf{H}$ is a homomorphism, define a **G**-**H** bibundle whose total space is:

$$X = M \times_{N,f_0,t} H \tag{15}$$

with the maps $\pi_1 : X \to M$ and $s : X \to N$.
The **G**-action comes from the obvious **G**-action on M, and the **H**-action is by composition.

This can be extended to give a bibundle for a Morita morphism $\mathbf{G} \leftarrow \mathbf{X} \to \mathbf{H}$ of topological groupoids. In **Diff**, at least the case of a Morita *equivalence* always works.

Some properties of bibundles are necessary because **Diff** does not have all pullbacks.:

Definition

A bundle E is **principal** when π is a surjective submersion and a quotient map (i.e. the action is free and transitive on fibres).
A bibundle is **regular** when it is principal as a (left) **G**-bundle, and the **H** action $\triangleright_H$ is a proper map (i.e. the preimage of a compact set is compact).

It is possible to *compose* regular bibundles.

Composition of Bibundles

If E is a regular $\mathbf{G}$-$\mathbf{H}$ bibundle, and E' a regular $\mathbf{H}$-$\mathbf{K}$ bibundle, there is a pullback over $\mathbf{H}$:

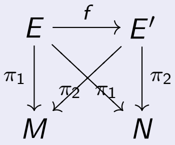
$$\begin{array}{ccccc} & & (E \times_N E') & & \\ & \swarrow & & \searrow & \\ & E & & E' & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ \mathbf{G} & & \mathbf{H} & & \mathbf{K} \end{array} \tag{16}$$

The pullback $E \times_N E'$ naturally becomes a $\mathbf{G}$-$\mathbf{H}$ bibundle through the actions of $\mathbf{H}$ on E and E'. It also has a diagonal action of $\mathbf{H}$ on it, by $h : (e, e') \mapsto (eh, h^{-1}e')$. The Hilsum-Skandalis tensor product for bibundles is then $E \otimes_{\mathbf{H}} E' = (E \times_N E')/\mathbf{H}$, as a $\mathbf{G}$-$\mathbf{K}$ bibundle. This composition respects the embedding of homomorphisms.

Definition

If **G** and **H** are Lie groupoids over M and N, define a category $\mathrm{BB}(G, H)$ with:

- *Objects*: regular $G - H$ bibundles
- *Morphisms*: For bibundles E and E', an arrow $f : E \to E'$ is a bundle map making



(17)

  commute, and which agrees with the **G** and **H**-actions.

Composition of two bibundles is by the Hilsum-Skandalis product.

(Note: In the topological case, the assumption of regularity isn't needed to define the analogous composition.)

The 2-Category of Lie Groupoids

We can assemble a 2-category **LG** of Lie groupoids.

Definition

The 2-category **LG** has Lie groupoids as objects, and for any **G** and **H**, there is a hom-category

$$\hom(\mathbf{G},\mathbf{H}) = BB(\mathbf{G},\mathbf{H}) \tag{18}$$

with horizontal composition by the HS tensor product

$$\otimes_{\mathbf{H}} : BB(\mathbf{G},\mathbf{H}) \times BB(\mathbf{H},\mathbf{K}) \rightarrow BB(\mathbf{G},\mathbf{K}) \tag{19}$$

Morita Equivalence Main Theorem

So we get the main result: the notion of equivalence in **LG** turns out to be the same as Morita equivalence.

Theorem

If two groupoids $\mathbf{G}_1$ and $\mathbf{G}_2$ are equivalent in **LG**, *then* $Rep(\mathbf{G}_1) \simeq Rep(\mathbf{G}_2)$.
This occurs exactly when there is a **G**-**H** *bibundle E which is left and right principal, and where both actions are proper. In this case, the inverse is $\overline{E}$, which is E seen as a* **H**-**G** *bibundle (using the inverse in* **G** *and* **H**).

In some settings, such as discrete groupoids, the converse is also true, but for Lie groupoids generally it is not.

Proof Idea

Given a bibundle $\mathbf{G}_1 \leftarrow E \rightarrow \mathbf{G}_2$, the functor

$$F = E \otimes_{\mathbf{G}_2} - : Rep(\mathbf{G}_2) \rightarrow Rep(\mathbf{G}_1) \tag{20}$$

and similarly

$$F' = E \otimes_{\mathbf{G}_1} - : Rep(\mathbf{G}_1) \rightarrow Rep(\mathbf{G}_2) \tag{21}$$

Then there are natural isomorphisms $F \circ F' \simeq 1_{Rep(\mathbf{G}_2)}$ and $F' \circ F \simeq 1_{Rep(\mathbf{G}_1)}$.