Toposes and Groupoids

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Introduction

- Theorem(Butz, Moerdijk '98): Let $\mathcal{E}$ be any topos with enough points. There exists a topological groupoid $\underline{G} = \{G_1 \rightrightarrows G_0\}$ for which there is an equivalence of topoi $\mathcal{E} \cong Sh(\underline{G})$.

Toposes (Topoi)

A (Grothendieck) *topos* is a category $\underline{\mathbf{T}}$ with:

1. finite limits and colimits,
2. exponential objects,
3. a subobject classifier Ω,
4. ‘bounded’ - Giraud’s theorem.

*This is an axiomatic description of a category given by sheaves on a site.

Examples

1. *Let X be a topological space, $\mathbf{Sh}(X)$ is a topos ($\Omega(U) = \{V | V \subset U\}$).
2. $\mathbf{Sets}$ is a topos with subobject classifier $\Omega = \mathbf{2}$.
3. Any presheaf category $[\mathbb{C}, \mathbf{Sets}]$ is a topos, $\Omega(C) = \mathbf{Sub}_{\hat{\mathbb{C}}}(Hom_{\mathbb{C}}(C, -))$; e.g., G-Sets.
4. A slice of a topos is a topos, internal presheaves on an internal category is a topos, etc....
   (The properties required of a topos are closed under a wide variety of categorical operations.)

Motivation

The idea is to think of $\mathbf{Sh}(X)$ as a categorical representation for the space X itself. Then we make arrow theoretic analogues of important properties/invariants of X.

These notions will carry verbatim to other topoi which are not spatial in nature.

Examples

- Cohomology becomes sheaf cohomology, i.e., consider the category of abelian group objects in $\mathbf{T}$, $A \in Ab(\mathbf{T})$, $H^n(\mathbf{T}, A) = R^n\Gamma(A)$. (Giraud's theorem implies we have enough injectives)
- In the category of étale schemes, with cover in the usual sense, the associated topos cohomology is étale cohomology
- Étale homotopy groups (pro-groups defined by limiting over the homotopy groups of certain simplicial sets, indexed by hypercovers, in a pointed, locally connected topos.)

Main Question

- How much do we enlarge our notion of topological space by looking at topoi?
- (Grothendieck-Galois Theory, SGAIV) In the category of étale schemes over a field, the associated topos is isomorphic to the category of G-Sets, for a pro-finite group G.

First Frames

- Definition: A *frame* is a distributive lattice with all joins and finite meets.
- Definition: A *morphism* of frames is a functor (as categories) that preserves finite meets (intersections) and infinite joins (unions).
- Definition: A 2-*morphism* is a natural transformation of functors.

Locales

- Defintion: The 2-category of locales $\underline{\underline{\mathbf{Loc}}} = \underline{\underline{\mathbf{Frm}}}^{op}$.
- Definition: A map f of locales is open iff corresponding map of frames has a left adjoint $f_!$ satisfying the Frobenius identity.

$$f_!(U \wedge f(V)) = f_!(V) \wedge U$$

- Note: A continuous map $f : X \to Y$, Y is T_1 separable, is open iff corresponding map of locales is open.

Spaces and locales

- Can define in $\underline{\underline{\mathbf{Loc}}}$ the notions points, embeddings, surjections, etc...
- Have adjoint functors $Loc \dashv pt : \underline{\underline{\mathbf{Loc}}} \to \mathbf{Top}$.
- Proposition: The adjunction restricts to an equivalence between the category of *sober* spaces and locales with *enough points*.
- Example: Any Hausdorff space is sober and can be recovered from its corresponding locale.

Locales and Topoi

- We can define the category of sheaves on a locale, as we do for spaces, giving an associated topos.

$$Sh : \underline{\underline{\mathbf{Loc}}} \to \underline{\underline{\mathbf{Top}}}$$

- Think of a locale as a generalized notion of 'space' intermediate between the notions of topological space and topoi.

- Theorem: A topos is localic iff it is generated by the subobjects of its terminal object.

The 2-category of Topoi

- A continuous map between topological spaces induces an adjoint pair on associated topoi, motivating the defintion of morphism.
- Definition: A (geometric) *morphism* of topoi $p : \mathcal{E} \to \mathcal{F}$, is an adjoint pair $p^* \dashv p_* : \mathcal{E} \to \mathcal{F},$ such that p^* preserves finite limits.
- A 2-*morphism* $p \Rightarrow q$ is a natural transformation $\eta : p^* \to q*$.

Geometric Morphisms

- A geometric morphism is $p : \mathcal{E} \to \mathcal{F}$

  1. *surjective* if p^* is faithful,
  2. *embedding* if p_* is full and faithful,
  3. *open* if for each $F \in \mathcal{F}$ the induced functor of posets $p_{F_*} : Sub_{\mathcal{E}}(p^*F) \to Sub_{\mathcal{F}}(F)$ has a left adjoint $(p_F)_!$.

- Note: A map of localic toposes is open iff corresponding map of locales is open.

Locales and Topoi (cont.)

- Theorem* (Localic Reflection)
  The 2-functor $sh : \underline{\underline{Loc}} \to \underline{\underline{Top}}$ induces an equivalence of categories
  $sh : \underline{Map}(X, Y) \to \underline{Hom}(Sh(X), Sh(Y))$
- The above functor has a left adjoint $Loc : \underline{\underline{Top}} \to \underline{\underline{Loc}}$ given by $\mathbf{T} \mapsto Sub_{\mathbf{T}}(1)$ (complete Heyting algebra)
- Thus localic spaces are embedded into topoi, i.e., we enlarged the category of 'spaces' without changing the underlying notion.

Covering Theorem

- Theorem (Diaconescu): For every (Grothendieck) topos $\mathcal{E}$ there exists a locale X and an open surjective (geometric) morphism $Sh(X) \twoheadrightarrow \mathcal{E}$.
- Corollary (Barr): For every Grothendieck topos $\mathcal{E}$ there exists a complete Boolean algebra B and a surjective geometric morphism $Sh(B) \twoheadrightarrow \mathcal{E}$.

Monads

Definition: A monad $\mathbf{T}$ in a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and natural transformations

$$\eta : 1_{\mathbf{C}} \to T \qquad \mu : T^2 \to T$$

satisfying the following identities

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "I"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "I"] \\
& T &
\end{tikzcd}$$

Main Example: Adjunctions

Algebras

- Definition: A T-*algebra* is an object $c \in \mathbf{C}$ and a map $h : T(c) \to c$ such that the following diagrams commute

$$\begin{tikzcd} T^2(c) \arrow[r, "Th"] \arrow[d, "\mu_c"'] & T(c) \arrow[d, "h"] \\ T(c) \arrow[r, "h"] & c \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "I"'] & T(c) \arrow[d, "h"] \\ & c \end{tikzcd}$$

- Definition: (Eilenberg-Moore Category) A *morphism* of T-algebras is a map $f : c \to c'$ such that the following diagram commutes

$$\begin{tikzcd} T(c) \arrow[r, "h"] \arrow[d, "Tf"'] & c \arrow[d, "f"] \\ T(c') \arrow[r, "h'"] & c' \end{tikzcd}$$

Tripleable

- Given an adjunction $L \dashv R : Y \to X$, the monad $T = RL$ on X induces a unique comparison functor $K : Y \to X^T$.

- The adjunction is *monadic* (tripleable) if the induced comparison functor is a categorical equivalence.

- Example: $F \dashv U : \mathbf{Grps} \to \mathbf{Sets}$

Beck's Theorem

Theorem: Given a functor $R : Y \to X$, then R is monadic if

1. R has a left adjoint $L : X \to Y$,
2. Y has coequalizers of reflexive pairs,
3. R preservers these coequalizers,
4. R reflects isomorphisms.

Beck-Chevalley Condition

- Assume the localic covering $Sh(X) \twoheadrightarrow \mathcal{E}$ is *essential*.

$$p_! \dashv p^* \dashv p_* : Sh(X) \to \mathcal{E}$$

- p^* is faithful, preserves colimits; thus, $\mathcal{E}$ is equivalent to the category of algebras for the monad $T = p^* p_!$.

- By the direct/inverse image adjunction, the category of T-algebras is equivalent to the category of coalgebras for the comonad $T' = p^* p_*$.

Descent

- Consider the (truncated) simplicial topos obtained by pulling back p^* along itself .

$$Sh(X) \times_{\mathcal{E}} Sh(X) \times_{\mathcal{E}} Sh(X) \rightarrow Sh(X) \times_{\mathcal{E}} Sh(X) \rightrightarrows Sh(X)$$

- An object $x \in Sh(X)$ satisfies *descent* if there exists a morphism $\theta : \pi_1^*(x) \rightarrow \pi_2^*(x)$ in $Sh(X) \times_{\mathcal{E}} Sh(X)$ such that $\Delta^*(\theta) = I_x$ and the following diagram in $Sh(X) \times_{\mathcal{E}} Sh(X) \times_{\mathcal{E}} Sh(X)$ commutes.

$$\begin{tikzcd}
\pi_1^*(x) \arrow[r, "\pi_{12}^*(\theta)"] \arrow[dr, "\pi_{13}^*(\theta)"'] & \pi_2^*(x) \arrow[d, "\pi_{23}^*(\theta)"] \\
& \pi_2^*(x)
\end{tikzcd}$$

Monadic Descent

- The Beck-Chevalley condition states an object satisfying descent $\theta : \pi_1^*(x) \to \pi_2^*(x)$ corresponds to a T'-coalgebra

$$\theta^{ad} : x \to \pi_{1*}\pi_2^*(x) = p^*p_*(x)$$

- The category of objects satisfying descent is equivalent to the category of T-algebras.

$$\mathfrak{Desc}(Sh(X)) \cong Sh(X)^T \cong \mathcal{E}$$

Lemmas

- Lemma: Open geometric morphisms are preserved by pullbacks.
- Lemma: The topoi $Sh(X) \times_{\mathcal{E}} Sh(X)$ and $Sh(X) \times_{\mathcal{E}} Sh(X) \times_{\mathcal{E}} Sh(X)$ are localic.
- Lemma: $Sh(X) \cong \acute{E}t(X)$

Reflection

Using localic reflection we can consider the simplicial topos

$$Sh(X) \times_{\mathcal{E}} Sh(X) \times_{\mathcal{E}} Sh(X) \to Sh(X) \times_{\mathcal{E}} Sh(X) \rightrightarrows Sh(X)$$

as defining a simplicial space in $\underline{\underline{Loc}}$.

$$G_2 \to G_1 \rightrightarrows G_0$$

This defines an internal category with $\acute{E}t(G_0) = Sh(X)$
Indeed, the twist isomorphism $G_1 \to G_1$ defines a (open) groupoid structure $\underline{G}$.

Groupoid Action

- We say an étale spaces over G_0 with an action of $\underline{G}$ is a $\underline{G}$-Set.
- This is an étale space $X \to G_0$ with a map $\theta : s^*(X) \to t^*(X)$ over G_1 such that $i*(\theta) = I_X$ and $\partial^0(\theta)\partial^2(\theta) = \partial^1(\theta)$.
- On the level of sheaves this proves

$$\underline{G}\text{-Sets} \cong \mathfrak{Desc}(Sh(X)) \cong \mathcal{E}$$

Morita Equivalence

- Theorem(Elephant C5.3.18): Let $\underline{G}$ and $\underline{H}$ be étale-complete open localic groupoids, and let $f : \underline{G} \to \underline{H}$ be a geometric morphism. Then there exists an étale-complete open groupoid $\underline{K}$ and groupoid morphisms g and h such that g is an open weak equivlanece, and such that the following diagram commutes up to isomorphism

$$\begin{tikzcd} \underline{K}\text{-Sets} \arrow[d, "g"'] \arrow[dr, "h"] & \\ \underline{G}\text{-Sets} \arrow[r, "f"] & \underline{H}\text{-Sets} \end{tikzcd}$$

Extensions

- Theorem(Moerdijk, Pronk '97): For any ringed topos $\mathcal{T} = (\mathbf{T}, \mathcal{O}_T)$ the following properties are equivalent:

  1. $\mathcal{T} \cong Sh(M)$ for some orbifold M (unique up to orbifold equivalence),
  2. $\mathcal{T} \cong Sh_L(M)$ for some manifold M and a compact Lie group L acting smoothly on X, so that the action has finite isotropy groups and faithful slice representations,
  3. $\mathcal{T}$ is an effective smooth etendue such that the diagonal $\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is a proper (topos) map,
  4. $\mathcal{T} \cong Sh(\underline{G})$ for some effective etale groupoid $\underline{G}$ such that $(s,t) : G_1 \to G_0 \times G_0$ is a proper map of spaces.

Applications

- Topological representation of sheaf cohomology
- fibered product of orbifolds
- change of base formulas for sheaf cohomology

Thank You